CHAPTER 3

THEORETICAL BACKGROUND

3.1 Smeared Crack Models

In order to develop an analysis method to solve problems involving cracking localization, the finite element method is selected as a tool for numerical calculation due to its powerful approximation. There are normally two main models proposed for consideration of cracking problems using the finite element method, i.e., discrete crack model and smeared crack model. The smeared crack model is selected in this study because the smeared crack model is suitable for problems with many simultaneous cracks that are inevitable in the cracking localization analysis.

In this study, the incremental formulation will be employed in the analysis. To begin the formulation, we consider the smeared crack model. The fundamental of the smeared crack model is the decomposition of the total strain increment $\Delta\boldsymbol{\varepsilon}$ into the strain increments of an uncracked solid part $\Delta\boldsymbol{\varepsilon}^{o}$ and a crack part $\Delta\boldsymbol{\varepsilon}^{cr}$ (De Borst and Nauta 1985) as

$$\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}^{o} + \Delta\boldsymbol{\varepsilon}^{cr}. \tag{3.1}$$

In the local coordinates, we have the local crack strain increment vector $\Delta\hat{\boldsymbol{\varepsilon}}^{cr}$ defined as

$$\Delta\hat{\boldsymbol{\varepsilon}}^{cr} = \begin{bmatrix}\Delta\hat{\varepsilon}_{nn}^{cr} & \Delta\hat{\gamma}_{nt}^{cr}\end{bmatrix}^{T} \tag{3.2}$$

in which $\Delta\hat{\varepsilon}_{nn}^{cr}$ and $\Delta\hat{\gamma}_{nt}^{cr}$ are the normal and shear crack strain increments, respectively. The relationship between the global crack strain increment vector $\Delta\boldsymbol{\varepsilon}^{cr}$ and the local crack strain increment vector $\Delta\hat{\boldsymbol{\varepsilon}}^{cr}$ is

$$\Delta\boldsymbol{\varepsilon}^{cr} = \mathbf{T}\Delta\hat{\boldsymbol{\varepsilon}}^{cr} \tag{3.3}$$

where $\mathbf{T}$ is a transformation matrix defined as

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ \sin^2\theta & \sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{3.4}$$

in which θ is the inclination angle of the normal of the crack. Similar to the local crack strain increment, we have the local traction increment $\Delta\hat{\mathbf{t}}^{cr}$, composed of normal crack traction increment $\Delta\hat{t}_{n}^{cr}$ and shear crack traction increment $\Delta\hat{s}_{t}^{cr}$, defined as

$$\Delta\hat{\mathbf{t}}^{cr} = \left[\Delta\hat{t}_n^{cr} \quad \Delta\hat{s}_t^{cr}\right] \tag{3.5}$$

which is related to the local crack strain increment vector $\Delta\hat{\varepsilon}^{cr}$ as

$$\Delta\hat{\mathbf{t}}^{cr} = \hat{\mathbf{D}}^{cr}\Delta\hat{\varepsilon}^{cr} \tag{3.6}$$

in which $\hat{\mathbf{D}}^{cr}$ is a crack constitutive matrix defined as

$$\hat{\mathbf{D}}^{cr} = \begin{bmatrix} D^I & 0 \\ 0 & D^{II} \end{bmatrix}. \tag{3.7}$$

Here, D^I and D^{II} are the mode I and II stiffness modulus, respectively. The local traction increment $\Delta\hat{\mathbf{t}}^{cr}$ is related to the global stress increment $\Delta\sigma$ as

$$\Delta\hat{\mathbf{t}}^{cr} = \mathbf{T}^T\Delta\sigma\,. \tag{3.8}$$

The constitutive relationship between the global stress increment $\Delta\sigma$ and the global strain increment of the uncracked solid $\Delta\varepsilon^o$ is given as

$$\Delta\sigma = \mathbf{D}^o\Delta\varepsilon^o \tag{3.9}$$

in which $\mathbf{D}^o$ is the constitutive matrix of the uncracked solid material.

Combining Eqs. (3.1)-(3.9), we obtain the overall stress-strain relationship (De Borst and Nauta 1985) as

$$\Delta\sigma = \left[\mathbf{D}^o - \mathbf{D}^o\mathbf{T}\left[\hat{\mathbf{D}}^{cr} + \mathbf{T}^T\mathbf{D}^o\mathbf{T}\right]^{-1}\mathbf{T}^T\mathbf{D}^o\right]\Delta\varepsilon = \overline{\mathbf{D}}\Delta\varepsilon \tag{3.10}$$

which is the constitutive law for the cracked material in the smeared crack model.

3.2 Characteristic Length in Smeared Crack Models

One of the important parameters in the smeared crack models is the characteristic length. The simple definition of the characteristic length is the width of the damaged strip where the crack constitutive law is satisfied. In the crack band model by Bazant and Oh (1983), the characteristic length or the crack band width is considered as one of material properties and it is considered to be dependent on the maximum aggregate size. This concept is not suitable for the finite element computation, especially for curved crack paths. The reason is that the boundary of the crack band may be different from the boundary of the existing elements. This will lead to different material properties within one element. Another idea is to use the boundary of elements as the boundary of the crack band instead of using a fixed crack band width. In this case, the size of the smeared crack band will depend on mesh size. Without careful consideration, the results of the computation will be mesh dependent.

The objectivity of the results can be achieved by modifying the constitutive law and making it depend on mesh size.

Oliver (1989) modeled a two-dimensional crack as a limiting case of two singular lines, which tend to coincide with each other. Across the two lines, the displacements are continuous but the displacement gradients are discontinuous (see Fig. 3.1). Following Oliver (1989), we define the characteristic length l^* as the ratio between the energy dissipated per unit surface area (fracture energy) and the energy dissipated per unit volume (specific energy) at a point, i.e.,

$$g_f = \frac{G_f}{l^*}. \tag{3.11}$$

where the energy dissipation represented by G_f and g_f is from time $t = 0$ to $t = \infty$.

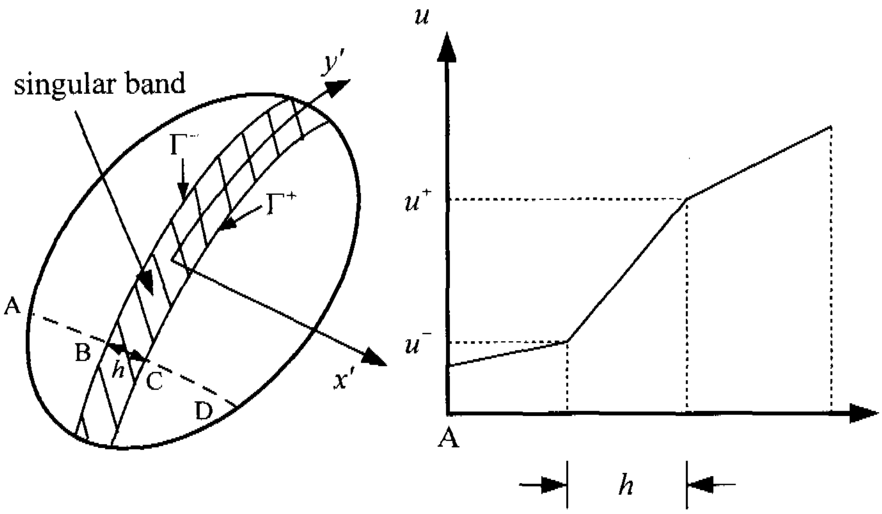

Fig. 3.1 A crack as a singular band between two singular lines (Oliver 1989)

Consider a crack band or a singular band, bounded by the two singular lines, in a solid with an infinitesimal width h (see Fig. 3.1). Along a coordinate line x', the displacement vector $\mathbf{u}$ may be expanded from its value in the Γ^- line as

$$\mathbf{u}(x', y') = \mathbf{u}^-(y') + \left(\frac{\partial \mathbf{u}}{\partial x'}\right)\Delta x' + O(h^2) \tag{3.12}$$

and, for a point in the Γ^+ line

$$\mathbf{u}^+(x', y') = \mathbf{u}^-(y') + \left(\frac{\partial \mathbf{u}}{\partial x'}\right)h(y') + O(h^2) \tag{3.13}$$

where the superscripts + and – denote values in the Γ^+ and Γ^- lines, respectively.

From Eqs. (3.12) and (3.13), we write

$$\mathbf{u}(x',y')=\mathbf{u}^-(y')+\frac{\Delta x'}{h}\left[\mathbf{u}^+(y')-\mathbf{u}^-(y')\right]+O(h^2)\approx \mathbf{u}^-(y')+\phi(x',y')\left[\mathbf{u}^+(y')-\mathbf{u}^-(y')\right] \quad (3.14)$$

where ϕ is a function that approximates $\frac{\Delta x'}{h}$ when $h \to 0$. It can be seen that

$$\phi = 0 \text{ on } \Gamma^- \text{ and } \phi = 1 \text{ on } \Gamma^+ . \quad (3.15)$$

By considering the equilibrium across the band and the energy dissipation within the band, it can be shown that (Oliver 1989)

$$g_f = G_f \frac{\partial \phi}{\partial x'} = \frac{G_f}{l^*} \quad (3.16)$$

where

$$l^*(x',y') = \left(\frac{\partial \phi}{\partial x'}\right)^{-1} . \quad (3.17)$$

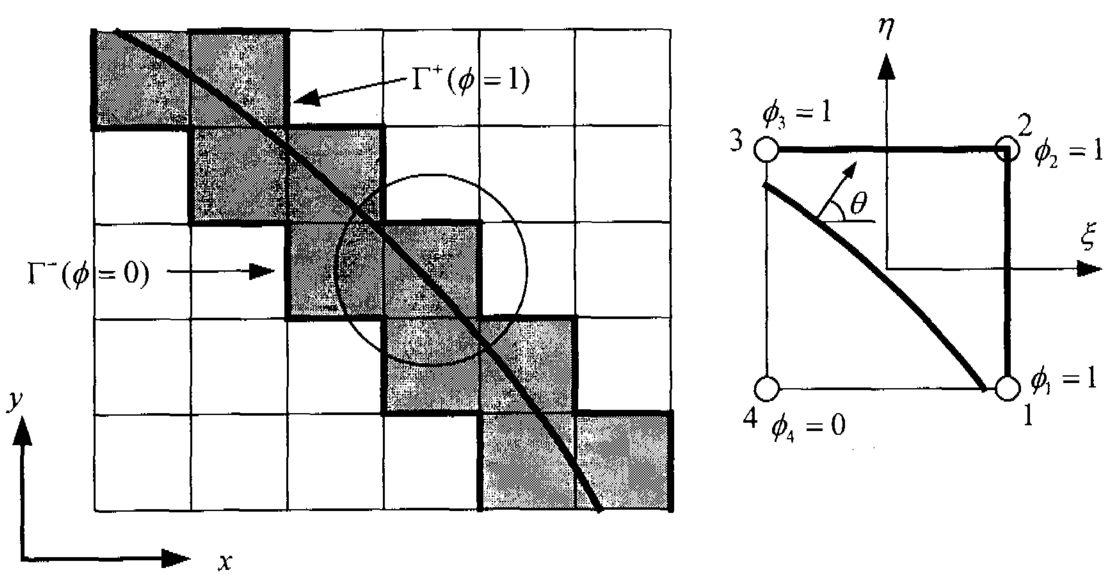

Fig. 3.2 A singular band in a finite element mesh

Consider a mesh of C^0 elements (Fig. 3.2). The function ϕ has to be a continuous, derivable function satisfying Eq. (3.15). Therefore, the following function defined in the natural coordinate system can be used, i.e.,

$$\phi(\xi,\eta) = \sum_{i=1}^{n_c} N_i^*(\xi,\eta)\phi_i \quad (3.18)$$

which n_c is the number of corner nodes of the elements, and N_i^* are the standard shape functions of an element of n_c nodes. Here, ϕ_i represents the value of the function ϕ at corner node i. As discussed earlier, a two-dimensional crack is modeled as a limiting case of two singular lines, which tend to coincide with each other. If the location of the crack within the element is known, ϕ_i will take the value +1 for each node on one side of the crack and 0 on the other side [Eq. (3.15)].

From Eqs. (3.17) and (3.18), we obtain the characteristic length as

$$l^*(\xi,\eta) = \left(\sum_{i=1}^{n_c} \left[\frac{\partial N_i^*(\xi,\eta)}{\partial x} \cos\theta + \frac{\partial N_i^*(\xi,\eta)}{\partial y} \sin\theta \right] \phi_i \right)^{-1} \tag{3.19}$$

where θ is the angle between the normal to the crack and the global x-axis, and ξ and η are the natural coordinates (see Fig. 3.2).

3.3 Localization and Bifurcation

Consider a system of a deformable body with cracks where the energy is dissipated. Following the concept by Nguyen (1987) and Brocca (1997), we define the total energy of the body as

$$U(u_i, \alpha_j) = U^M(u_k, \alpha_l) + U^D(\alpha_m) \tag{3.20}$$

where $U^M(u_k, \alpha_l)$ is the mechanical potential energy and and $U^D(\alpha_m)$ is the dissipated energy. The arguments of the function u_i (i = 1, 2,…,N) and α_j (j = 1,2,…,K) represent the reversible variables and irreversible variables, respectively. Here, N is the number of the reversible variables and K is the number of the irreversible variables.

The mechanical potential energy is the sum of the strain energy and the external potential energy. The dissipated energy is the energy that is transformed from mechanical energy to thermal energy through irreversible processes.

Applying the stationary conditions to Eq. (3.20), we obtain

$$\frac{\partial U}{\partial u_i} = 0\, , \frac{\partial U}{\partial \alpha_j} = 0\, , i = 1,2,\ldots\ldots,N\, ;\, j = 1,2,\ldots.K\, . \tag{3.21}$$

From Eq. (3.21), the equilibrated solution can be obtained. Employing the obtained solution, we can express the reversible parameters in terms of the irreversible parameters, i.e., $u_i = u_i(\alpha_j)$. Therefore, we can express the total energy in Eq. (3.20) as a function of only the irreversible parameters, i.e.,

$$U^*(\alpha_l) = U^{*M}(\alpha_j) + U^D(\alpha_m) \tag{3.22}$$

where $U^*(\alpha_l) = U(u_i(\alpha_k), \alpha_j)$ and $U^{*M}(\alpha_j) = U^M(u_k(\alpha_m), \alpha_l)$.

The sign of the eigenvalues of the Hessian Matrix $\left[\dfrac{\partial^2 U}{\partial \alpha_i \partial \alpha_j}\right]$ are used to check the stability of the equilibrated solution obtained form Eq. (3.21). If all the eigenvalues are positive, the equilibrated solution is stable and there is no bifurcation. Otherwise, the solution is unstable and the bifurcation, which leads to the localization, occurs.

3.4 Illustrative Models

Following Brocca (1997), we consider some illustrative models to understand the nature of the localization problems.

3.4.1 Model with a spring and a single crack element

Consider the model in Fig. 3.3 that consists of a spring with constant k and a crack element.

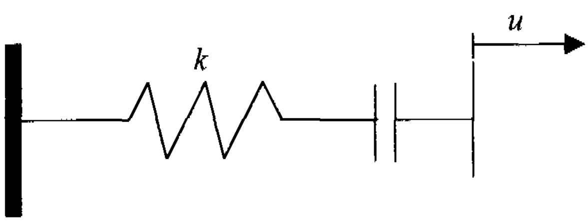

Fig. 3.3 Model with a spring and a single crack element

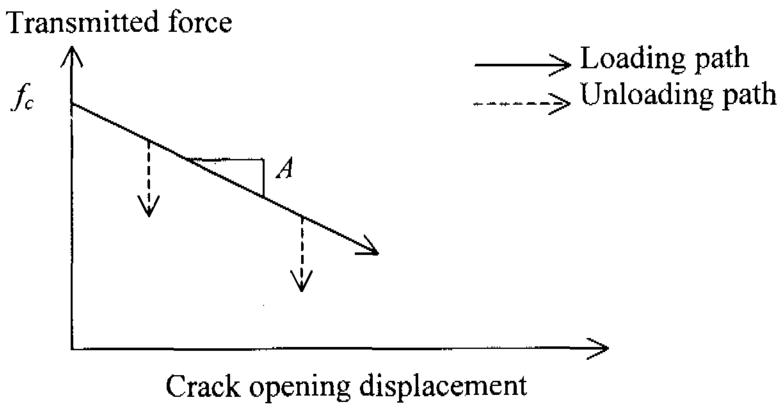

Fig. 3.4 Behavior of crack element

The crack element starts to open when the force f reaches the capacity f_c and after that the force changes as a linear function of the crack opening displacement α with modulus A shown in Fig. 3.4. The crack opening displacement is assumed to be a monotonically increasing function and its increment must be non-negative. When A is positive and the crack element shows a softening behavior, the crack element has two choices with a decreasing force, i.e., following the solid line called loading path or the dashed line called unloading path in Fig. 3.4. The problem is to find α for a given displacement u that is assumed to be monotonically increasing. Therefore, it is not necessary to consider the unloading path in this problem.

The equilibrium condition of the problem can be written as

$$f_c - A\alpha = k(u-\alpha) \tag{3.23}$$

which gives the solution as

$$\alpha = \frac{f_c - ku}{k-A}. \tag{3.24}$$

An equivalent formulation is possible by considering a potential function U defined as

$$U = \frac{1}{2}k(u-\alpha)^2 + f_c\alpha - \frac{1}{2}A\alpha^2. \tag{3.25}$$

Applying $\frac{dU}{d\alpha} = 0$, we obtain

$$\alpha = \frac{f_c - ku}{k-A}. \tag{3.26}$$

The above two formulations are equivalent, but the variational formulation provides benefits for judgement of the stability of the solution. The stability of the solution is defined by the sign of the work done by the external system for any disturbance given to the equilibrium solution. It can be shown that the stability of the equilibrium solution is judged by the sign of the second derivative of the potential function U with respect to the crack opening displacement, i.e.,

$$\frac{d^2U}{d\alpha^2} = k - A \qquad \text{if} \quad \begin{array}{l} > 0 : \text{stable } (k>A) \\ < 0 : \text{unstable } (k<A). \end{array} \tag{3.27}$$

When $A<k$, the solution in Eq. (3.27) is stable, and the loading path in Fig. 3.4 is followed with the increasing value of u. On the other hand, when $A>k$, the solution is unstable. Hence, in this case, when u is increased from 0, the crack element starts to open at $u = \frac{f_c}{k}$, and the crack opening displacement increases in an unstable manner.

The total energy U in Eq. (3.25) is considered as the sum of the mechanical potential energy U^M and the dissipated energy U^D. The mechanical energy is reversible while the dissipated energy is irreversible as it is dissipated from the system in form of heat. In this example, the mechanical potential energy is the strain energy stored in the spring and the dissipated energy is the energy dissipated at the crack element, i.e.,

$$U^M = \frac{1}{2}k(u-\alpha)^2 \qquad \text{and} \qquad U^D = f_c\alpha - \frac{1}{2}k\alpha^2. \tag{3.28}$$

3.4.2 Model with two crack elements

The model shown in Fig. 3.5 consists of two cracks without spring. The potential energy is given as the sum of the dissipated energy of the two crack elements.

$$U = f_c\alpha - \frac{1}{2}A\alpha^2 + f_c(u-\alpha) - \frac{1}{2}A(u-\alpha)^2 \tag{3.29}$$

where α and u-α are crack opening displacements of the two crack elements.

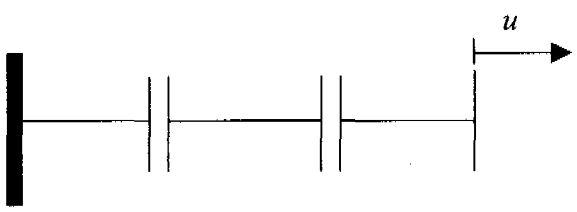

Fig. 3.5 Model with two crack elements

The stationary condition leads to the fundamental solution, i.e.,

$$\frac{dU}{d\alpha} = A(u-2\alpha) = 0 \quad \rightarrow \quad \alpha = \frac{1}{2}u. \tag{3.30}$$

The stability of the solution can be checked by taking the second derivative of the potential function U with respect to α as

$$\frac{d^2U}{d\alpha^2} = -2A \quad \text{if} \quad \begin{array}{l} >0 : \text{stable} \\ <0 : \text{unstable} \end{array} \tag{3.31}$$

If the stationary solution is stable with $A<0$, α is equal to $u/2$ and the crack opening displacements of the two crack elements are equal. When the stationary solution is unstable with $A>0$, the crack opening displacements of the two crack elements can no longer be equal.

When the solution is unstable, the localized solution is determined by seeking for a minimum point of total potential energy under the condition that the crack opening displacements of the two crack elements are nonnegative. The localized solution in this problem is either $\alpha = 0$ or $\alpha = u$. Both of them minimize the total potential energy within the admissible values for α, $0 \le \alpha \le u$, as shown in Fig. 3.6.

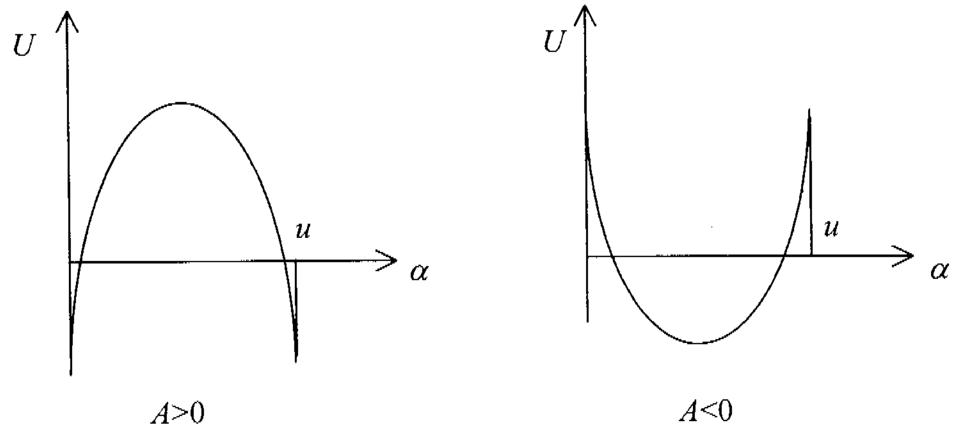

Fig. 3.6 Variation of total energy for a model with two crack elements

3.4.3 Model with a spring and two crack elements

Consider Fig. 3.7. The total potential energy of this system is given by

$$U = f_c \alpha_1 - \frac{1}{2} A \alpha_1^2 + f_c \alpha_2 - \frac{1}{2} A \alpha_2^2 + \frac{1}{2} k (u - \alpha_1 - \alpha_2)^2 \qquad (3.32)$$

where α_1 and α_2 are crack opening displacements of the two crack elements.

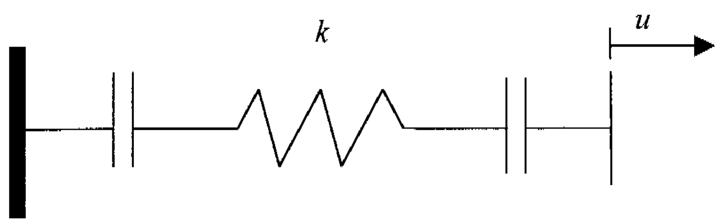

Fig. 3.7 Model with a spring and two crack elements

The stationary condition leads to

$$\frac{\partial U}{\partial \alpha_1} = f_c - A\alpha_1 - k(u - \alpha_1 - \alpha_2) = 0,$$
$$\frac{\partial U}{\partial \alpha_2} = f_c - A\alpha_2 - k(u - \alpha_1 - \alpha_2) = 0. \qquad (3.33)$$

The stability can be examined by the sign of the eigenvalues of the Hessian matrix whose components are the second derivative of the total energy with respect to the crack opening displacement, i.e.,

$$\left[\frac{\partial^2 U}{\partial \alpha_i \partial \alpha_j}\right] = \begin{bmatrix} -A+k & k \\ k & -A+k \end{bmatrix}. \tag{3.34}$$

The eigenvalues of the Hessian are found to be $-A$ and $-A+k$. They are the principal curvatures of the energy surface which represents the value of the total potential energy U as a function of α_i's. From the eigenvalues, there are three consideration cases.

Case 1 If $A<0$, both of the eigenvalues are positive. There is no localization in this case. The stationary solutions are stable, and the crack opening displacements of the two crack elements are both $\frac{(f_c - ku)}{(A-2k)}$. It is clearer to present the conclusion for this case by using the plot of the energy surface function in terms of the crack opening displacements as shown in Fig. 3.8. The vertical axis represents the energy function U, and the horizontal axes represent the crack opening displacements, α_1 and α_2. The black point in the figure illustrates the stable equilibrium point for this case, and the two crack opening displacements are $\frac{(f_c - ku)}{(A-2k)}$.

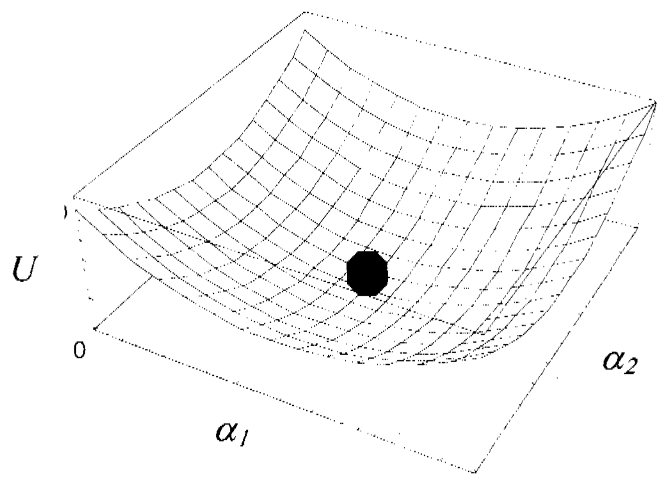

Fig. 3.8 Variation of total energy for a model with a spring and two crack elements

Case 2 If $2k>A>0$, one of the two eigenvalues will be negative. The stationary solution is unstable. In the other words, only one of the two crack elements continues to open and the other undergoes the elastic unloading. It is found that $(\alpha_1, \alpha_2) = ((f_c - ku)/(A-k), 0)$ or $(\alpha_1, \alpha_2) = (0, (f_c - ku)/(A-k))$ minimizes energy surface in the region where the crack opening displacements are within the admissible values. The energy surface of this case is shown in Fig. 3.9. The black point shows the unstable equilibrium state. The arrows show the directions that the system tries to search for a stable solution. It is noted that there are two possible solutions, i.e., either α_1 or α_2 is equal to zero.

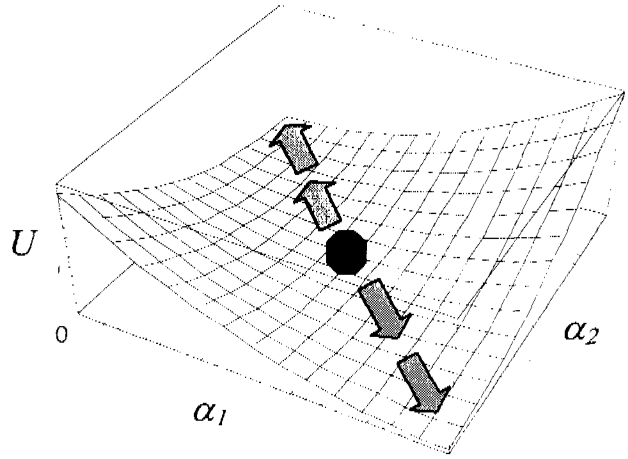

Fig. 3.9 Variation of total energy for a model with a spring and two crack elements

Case 3 If $A>2k$, both of the eigenvalues are negative. The stationary solution is unstable, and the cracks unstably open when the applied force reaches f_c. The plot of the energy surface is shown in Fig. 3.10.

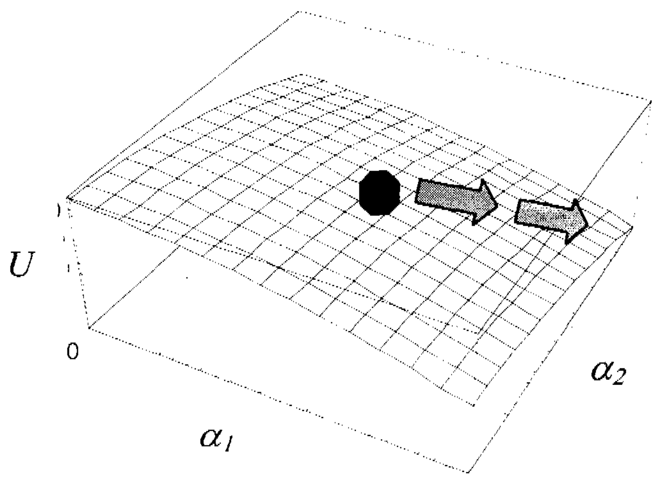

Fig. 3.10 Variation of total energy for a model with a spring and two crack elements