CHAPTER 4

SMEARED CRACK FINITE ELEMENT ANALYSIS FOR CRACKING LOCALIZATION

In this chapter, the derivation of the proposed approach for the analysis of the cracking localization using the smeared crack model is described. The fundamental scheme of the smeared crack model is the decomposition of the total strain increment $\Delta\varepsilon$ into the strain increment of the uncracked solid part $\Delta\varepsilon^{o}$ and the strain increment of the crack part $\Delta\varepsilon^{cr}$, i.e.,

$$\Delta\varepsilon \;=\; \Delta\varepsilon^{o} + \Delta\varepsilon^{cr}. \tag{4.1}$$

In the analysis of the cracking localization, the consideration of the stability and bifurcation of equilibrium state has to be done. To this end, the total energy increment of the system in terms of the irreversible parameters is required. The total energy increment can be written as

$$\begin{aligned} \Delta U &= \Delta U^{M} + \Delta U^{D} \\ &= \left[\frac{1}{2}\int_{V} \Delta\varepsilon^{o^{T}} \Delta\sigma dV - \int_{V} \Delta\mathbf{u}^{T} \Delta\mathbf{f} dV - \int_{S} \Delta\mathbf{u}^{T} \Delta\mathbf{t} dS\right] + \left[\frac{1}{2}\int_{V} \Delta\hat{\varepsilon}^{cr^{T}} \Delta\hat{\mathbf{t}}^{cr} dV\right] \end{aligned} \tag{4.2}$$

where ΔU^{M} and ΔU^{D} represent the mechanical potential energy increment and the dissipated energy increment, respectively. Here, $\Delta\mathbf{t}$ and $\Delta\mathbf{f}$ denote the surface traction increment vector and the body force increment vector, respectively. By substituting Eqs. (3.6) and (3.9) into Eq. (4.2), we get

$$\begin{aligned} \Delta U &= \Delta U^{M} + \Delta U^{D} \\ &= \left[\frac{1}{2}\int_{V} \Delta\varepsilon^{o^{T}} \mathbf{D}^{o} \Delta\varepsilon^{o} dV - \int_{V} \Delta\mathbf{u}^{T} \Delta\mathbf{f} dV - \int_{S} \Delta\mathbf{u}^{T} \Delta\mathbf{t} dS\right] + \left[\frac{1}{2}\int_{V} \Delta\hat{\varepsilon}^{cr^{T}} \hat{\mathbf{D}}^{cr} \Delta\hat{\varepsilon}^{cr} dV\right]. \end{aligned} \tag{4.3}$$

From Eq. (4.3), it can be seen that the dissipated energy is presented in terms of the local crack strain increment. In the consideration of the stationary condition of the system, it is more suitable to transform the local crack strain increment into the global crack strain increment by using the relationship between the local crack strain increment $\Delta\hat{\varepsilon}^{cr}$ and the global crack strain increment $\Delta\varepsilon^{cr}$ expressed as

$$\Delta\hat{\varepsilon}^{cr} = \hat{\mathbf{T}}\Delta\varepsilon^{cr} \tag{4.4}$$

in which $\hat{\mathbf{T}}$ is a 2×3 matrix defined as

$$\hat{\mathbf{T}} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{4.5}$$

where θ is the inclination angle of the normal of the crack.

Using Eq. (4.4) in Eq. (4.3), we get

$$\Delta U = \Delta U^M + \Delta U^D$$
$$= \left[\frac{1}{2}\int_V \Delta\varepsilon^{o^T} \mathbf{D}^o \Delta\varepsilon^o dV - \int_V \Delta\mathbf{u}^T \Delta\mathbf{f} dV - \int_S \Delta\mathbf{u}^T \Delta\mathbf{t} dS\right] + \left[\frac{1}{2}\int_V \Delta\varepsilon^{cr^T} \mathbf{D}^{cr} \Delta\varepsilon^{cr} dV\right] \quad (4.6)$$

in which $\mathbf{D}^{cr}$ is a matrix defined as

$$\mathbf{D}^{cr} = \hat{\mathbf{T}}^T \hat{\mathbf{D}}^{cr} \hat{\mathbf{T}}. \quad (4.7)$$

In the expression for the total energy increment in Eq. (4.6), the irreversible variable that has to be considered in the stability analysis is the crack strain increment $\Delta\varepsilon^{cr}$. The first variation of the total energy with respect to this crack strain increment results in the equilibrium path. The second variation will give the information in the stability condition of the obtained equilibrium path. Since the total energy increment is a functional of the crack strain increment function, the calculus of variations is required. To avoid this difficulty, we introduce a crack displacement increment vector $\Delta\mathbf{u}^{cr}$ defined as

$$\Delta\mathbf{u} = \Delta\mathbf{u}^o + \Delta\mathbf{u}^{cr} \quad (4.8)$$

in which the strain increments computed separately from $\Delta\mathbf{u}$, $\Delta\mathbf{u}^o$, and $\Delta\mathbf{u}^{cr}$ are $\Delta\varepsilon$, $\Delta\varepsilon^o$, and $\Delta\varepsilon^{cr}$, respectively.

Consider the i^{th} element in the finite element analysis. The element is assumed to be a cracked element. Interpolate these three displacement increments from nodal quantities, i.e.,

$$\Delta^i\mathbf{u} = \mathbf{N}\Delta^i\mathbf{U} \quad (4.9a)$$
$$\Delta^i\mathbf{u}^o = \mathbf{N}\Delta^i\mathbf{U}^o \quad (4.9b)$$
$$\Delta^i\mathbf{u}^{cr} = \mathbf{N}\Delta^i\mathbf{U}^{cr} \quad (4.9c)$$

where

$$\Delta^i\mathbf{U} = \Delta^i\mathbf{U}^o + \Delta^i\mathbf{U}^{cr} \quad (4.10)$$

in which $\Delta^i\mathbf{U}$, $\Delta^i\mathbf{U}^o$, and $\Delta^i\mathbf{U}^{cr}$ are the nodal quantities of $\Delta\mathbf{u}$, $\Delta\mathbf{u}^o$, and $\Delta\mathbf{u}^{cr}$, respectively. Here, $\mathbf{N}$ is the shape function matrix. Note that the superscript i for the i^{th} element is used in the equation because the nodal crack displacement increments of the same node for different elements can be different. The reason is that the crack displacement increment is defined so that its strain increment represents the crack strain increment. The continuity of the crack displacement increment between elements is not required and must not be enforced. One clear example is a problem with one cracked element surrounded by uncracked elements. In the cracked element including its boundary, non-zero crack displacement increment can be expected.

However, in the surrounding uncracked elements, the crack displacement increment is expected to be zero because there is no crack in those elements. On the contrary, the total displacement increment $\Delta\mathbf{u}$ must be continuous across elements. Therefore, the superscript i representing the element number is not actually necessary for the nodal values of the total displacement increment. Similar to the crack displacement increment, the displacement increment related to the strain increment of the uncracked solid material $\Delta^i\mathbf{u}^o$ is not continuous across the boundaries of the element. Therefore, the superscript i is required.

Computing strains from Eq. (4.9), we obtain Eq. (4.1), i.e.,

$$\Delta^i\varepsilon = \Delta^i\varepsilon^o + \Delta^i\varepsilon^{cr} \tag{4.11}$$

where

$$\Delta^i\varepsilon = \mathbf{B}\Delta^i\mathbf{U} \tag{4.12a}$$

$$\Delta^i\varepsilon^o = \mathbf{B}\Delta^i\mathbf{U}^o \tag{4.12b}$$

$$\Delta^i\varepsilon^{cr} = \mathbf{B}\Delta^i\mathbf{U}^{cr}. \tag{4.12c}$$

The stress increment can be derived from Eqs. (3.9), (3.10), (4.12a), and (4.12b) as

$$\Delta\sigma = \overline{\mathbf{D}}\mathbf{B}\Delta^i\mathbf{U} = \mathbf{D}^o\mathbf{B}\Delta^i\mathbf{U}^o. \tag{4.13}$$

From Eq. (4.13), it is observed that if the constitutive matrix for the uncracked solid material $\mathbf{D}^o$ is assumed to be constant within the element, the smeared constitutive matrix $\overline{\mathbf{D}}$ must be also constant within the element. Otherwise, the consistency in the order of the stress increment will not be obtained. If the characteristic length l^* in Eq. (3.19) is directly employed in the formulation, the smeared constitutive matrix $\overline{\mathbf{D}}$ will not be a constant but a function of position. This is because $\overline{\mathbf{D}}$ is a function of l^* which is a function of position as shown in Eq. (3.19).

To avoid this problem, the average value of the characteristic length l^* in Eq. (3.19) will be used in the element. Keeping the dissipated energy from the element from time $t = 0$ to $t = \infty$ unchanged, we can write

$$W^D = \int_V g_f dV = \int_V \frac{G_f}{l^*} dV = \int_V \frac{G_f}{\bar{l}^*} dV \tag{4.14}$$

in which W^D is the energy dissipated from the element from time $t = 0$ to $t = \infty$. Here, $\bar{l}^*$ is the average characteristic length and V is the volume of the element. Since the fracture energy G_f is a material property, we get

$$G_f \int_V \frac{1}{l^*} dV = \frac{G_f}{\bar{l}^*} \int_V dV \tag{4.15}$$

which yields

$$\bar{l}^* = \left(\frac{1}{V} \int_V \frac{1}{l^*} dV \right)^{-1} . \tag{4.16}$$

By substituting the characteristic length l^* given in Eq. (3.19) into Eq. (4.16), the average characteristic length $\bar{l}^*$ can be evaluated.

Substituting Eq. (4.11) into Eq. (4.6), we get the total energy increment for the i^{th} element as

$$\Delta U = \frac{1}{2} \int_V \left(\Delta^i \varepsilon - \Delta^i \varepsilon^{cr}\right)^T \mathbf{D}^o \left(\Delta^i \varepsilon - \Delta^i \varepsilon^{cr}\right) dV + \frac{1}{2} \int_V \Delta^i \varepsilon^{cr^T} \mathbf{D}^{cr} \Delta^i \varepsilon^{cr} dV - \int_V \Delta^i \mathbf{u}^T \Delta \mathbf{f} dV - \int_S \Delta^i \mathbf{u}^T \Delta \mathbf{t} dS \tag{4.17}$$

which yields

$$\begin{aligned} \Delta U = {} & \frac{1}{2} \int_V \Delta^i \varepsilon^T \mathbf{D}^o \Delta^i \varepsilon dV - \frac{1}{2} \int_V \Delta^i \varepsilon^T \mathbf{D}^o \Delta^i \varepsilon^{cr} dV - \frac{1}{2} \int_V \Delta^i \varepsilon^{cr^T} \mathbf{D}^o \Delta^i \varepsilon dV + \\ & \frac{1}{2} \int_V \Delta^i \varepsilon^{cr^T} \mathbf{D}^o \Delta^i \varepsilon^{cr} dV + \frac{1}{2} \int_V \Delta^i \varepsilon^{cr^T} \mathbf{D}^{cr} \Delta^i \varepsilon^{cr} dV - \int_V \Delta^i \mathbf{u}^T \Delta \mathbf{f} dV - \int_S \Delta^i \mathbf{u}^T \Delta \mathbf{t} dS. \end{aligned} \tag{4.18}$$

Substituting Eqs. (4.9a), (4.12a) and (4.12c) into Eq. (4.18), we get

$$\begin{aligned} \Delta U = {} & \frac{1}{2} \int_V \Delta^i \mathbf{U}^T \mathbf{B}^T \mathbf{D}^o \mathbf{B} \Delta^i \mathbf{U} dV - \frac{1}{2} \int_V \Delta^i \mathbf{U}^T \mathbf{B}^T \mathbf{D}^o \mathbf{B} \Delta^i \mathbf{U}^{cr} dV - \frac{1}{2} \int_V \Delta^i \mathbf{U}^{cr^T} \mathbf{B}^T \mathbf{D}^{cr} \mathbf{B} \Delta^i \mathbf{U} dV + \\ & \frac{1}{2} \int_V \Delta^i \mathbf{U}^{cr^T} \mathbf{B}^T \mathbf{D}^o \mathbf{B} \Delta^i \mathbf{U}^{cr} dV + \frac{1}{2} \int_V \Delta^i \mathbf{U}^{cr^T} \mathbf{B}^T \mathbf{D}^{cr} \mathbf{B} \Delta^i \mathbf{U}^{cr} dV - \\ & \int_V \Delta^i \mathbf{U}^T \mathbf{N}^T \Delta \mathbf{f} dV - \int_S \Delta^i \mathbf{U}^T \mathbf{N}^T \Delta \mathbf{t} dS. \end{aligned} \tag{4.19}$$

Applying the stationary condition $\delta(\Delta U) = 0$ and assuming that $\mathbf{D}^o$ and $\mathbf{D}^{cr}$ are symmetric, we get

$$\begin{aligned} \delta(\Delta U) = {} & \delta(\Delta^i \mathbf{U}^T) \int_V \mathbf{B}^T \mathbf{D}^o \mathbf{B} dV \Delta^i \mathbf{U} - \delta(\Delta^i \mathbf{U}^T) \int_V \mathbf{B}^T \mathbf{D}^o \mathbf{B} dV \Delta^i \mathbf{U}^{cr} - \delta(\Delta^i \mathbf{U}^{cr^T}) \int_V \mathbf{B}^T \mathbf{D}^o \mathbf{B} dV \Delta^i \mathbf{U} \\ & + \delta(\Delta^i \mathbf{U}^{cr^T}) \int_V \mathbf{B}^T \mathbf{D}^o \mathbf{B} dV \Delta^i \mathbf{U}^{cr} + \delta(\Delta^i \mathbf{U}^{cr^T}) \int_V \mathbf{B}^T \mathbf{D}^{cr} \mathbf{B} dV \Delta^i \mathbf{U}^{cr} \\ & - \delta(\Delta^i \mathbf{U}^T) \int_V \mathbf{N}^T \Delta \mathbf{f} dV - \delta(\Delta^i \mathbf{U}^T) \int_S \mathbf{N}^T \Delta \mathbf{t} dS = 0. \end{aligned} \tag{4.20}$$

Since $\delta(\Delta \mathbf{U}^T)$ and $\delta(\Delta^i \mathbf{U}_{cr}^T)$ are arbitrary, we obtain

$$\begin{bmatrix} \int_V \mathbf{B}^T \mathbf{D}^e \mathbf{B} dV & -\int_V \mathbf{B}^T \mathbf{D}^e \mathbf{B} dV \\ -\int_V \mathbf{B}^T \mathbf{D}^e \mathbf{B} dV & \int_V \mathbf{B}^T \mathbf{D}^e \mathbf{B} dV + \int_V \mathbf{B}^T \mathbf{D}^{cr} \mathbf{B} dV \end{bmatrix} \begin{Bmatrix} \Delta^i \mathbf{U} \\ \Delta^i \mathbf{U}^{cr} \end{Bmatrix} = \begin{Bmatrix} \int_V \mathbf{N}^T \Delta \mathbf{f} dV + \int_S \mathbf{N}^T \Delta \mathbf{t} dS \\ \mathbf{0} \end{Bmatrix}. \tag{4.21}$$

which is the element stiffness equation for the i^{th} element. After assembling all element stiffness equations and applying prescribed displacements and forces, we arrange the global stiffness equation as

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{U} \\ \Delta \mathbf{U}^{cr} \end{Bmatrix} = \begin{Bmatrix} \Delta \mathbf{R}_1 \\ \Delta \mathbf{R}_2 \end{Bmatrix}. \tag{4.22}$$

The static condensation is used to remove the nodal total displacement increment from the obtained global matrix equation. Therefore, the equation can be written in the following form, i.e.,

$$\mathbf{K}^{cr} \Delta \mathbf{U}^{cr} = \Delta \mathbf{R}^{cr} \tag{4.23}$$

where $\mathbf{K}^{cr}$ and $\Delta \mathbf{R}^{cr}$ are defined as

$$\begin{aligned} \mathbf{K}^{cr} &= \mathbf{K}_{22} - \mathbf{K}_{21} \mathbf{K}_{11}^{-1} \mathbf{K}_{12}, \\ \Delta \mathbf{R}^{cr} &= \Delta \mathbf{R}_2 - \mathbf{K}_{21} \mathbf{K}_{11}^{-1} \Delta \mathbf{R}_1. \end{aligned} \tag{4.24}$$

In the consideration of the stability condition, the eigenvalue analysis of $\mathbf{K}^{cr}$ is performed. However, it must be noted that Eq. (4.23) is a singular equation because $\Delta \mathbf{U}^{cr}$ contains rigid-body crack displacement increments. These rigid-body crack displacement increments will result in zero eigenvalues of the stiffness $\mathbf{K}^{cr}$. When the numerical eigenvalue analysis is performed on the stiffness matrix, we may not obtain zero eigenvalues for these rigid-body crack displacement increments but very small numbers, instead. Therefore, the result will be indistinguishable from those modes with real small non-zero eigenvalues. To avoid this confusion, constraints to remove the rigid-body crack displacement increments must be applied to the equation.

Let us consider Fig. 4.1a. It is a 2-noded bar element with the linear interpolation function. Here, ΔU_1^{cr} and ΔU_2^{cr} are the nodal crack displacement increments at node 1 and 2, respectively. The crack strain increment is the slope of a line connecting 2 nodal crack displacement increments. If the crack strain increment is kept constant, the nodal crack displacement increments can be any value, given that the slope of the line connecting the 2 nodal crack displacement increments is the same. In the other words, the line connecting these 2 nodal crack displacement increments can be freely moved up or down without changing its slope. This kind of crack displacement increment is the rigid-body crack displacement increment.

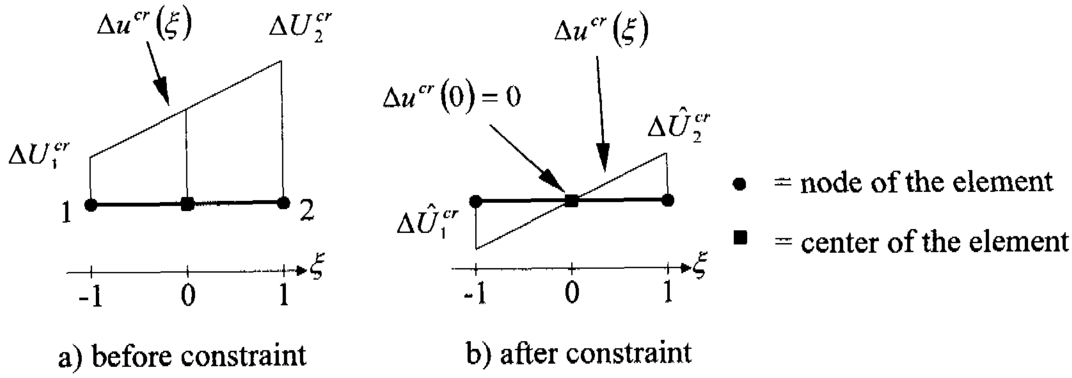

Fig. 4.1 Schematic illustration of constraint to remove rigid-body crack displacement for one-dimensional bar element

To remove the rigid-body crack displacement increment in Fig. 4.1a, we select to set the crack displacement increment at the center of the element to be zero. In the natural coordinate, the crack displacement increment is expressed as

$$\Delta u^{cr}(\xi)=\frac{1-\xi}{2}\Delta U_1^{cr}+\frac{1+\xi}{2}\Delta U_2^{cr} \tag{4.25}$$

where ξ is the natural coordinate ranging from -1 to 1.

Setting the crack displacement increment at the center of the element to zero gives

$$\Delta u^{cr}(0)=0=\frac{\Delta U_1^{cr}}{2}+\frac{\Delta U_2^{cr}}{2} \tag{4.26}$$

which yields the constraint equation for one-dimensional bar problem, i.e.,

$$\Delta U_1^{cr}+\Delta U_2^{cr}=0\,. \tag{4.27}$$

Constraining the rigid-body crack displacement increments can be done in this fashion because the magnitudes of the crack displacement increments are not important. The important things are the crack strain increments. As long as the values of the crack strain increments are not constrained, the generality is not lost.

For two-dimensional problems, $\Delta \mathbf{U}^{cr}$ also contains the rigid-body crack displacement increments, i.e., two translations and one rotation. To avoid these rigid-body displacement increments, the constraints must be also used. Consider the four-noded qualilateral element in Fig 4.2. The element employs the bilinear interpolation function. The crack displacement increments are expressed as

$$\Delta u^{cr}(\xi,\zeta)=\frac{(1-\xi)(1-\zeta)}{4}\Delta U_1^{cr}+\frac{(1+\xi)(1-\zeta)}{4}\Delta U_2^{cr}+\frac{(1+\xi)(1+\zeta)}{4}\Delta U_3^{cr}+\frac{(1-\xi)(1+\zeta)}{4}\Delta U_4^{cr}$$
$$\Delta v^{cr}(\xi,\zeta)=\frac{(1-\xi)(1-\zeta)}{4}\Delta V_1^{cr}+\frac{(1+\xi)(1-\zeta)}{4}\Delta V_2^{cr}+\frac{(1+\xi)(1+\zeta)}{4}\Delta V_3^{cr}+\frac{(1-\xi)(1+\zeta)}{4}\Delta V_4^{cr} \tag{4.28}$$

in which ξ and ζ are the two natural coordinates. Here, Δu^{cr} and Δv^{cr} represent the crack displacement increments in the global x- and y- directions, respectively.

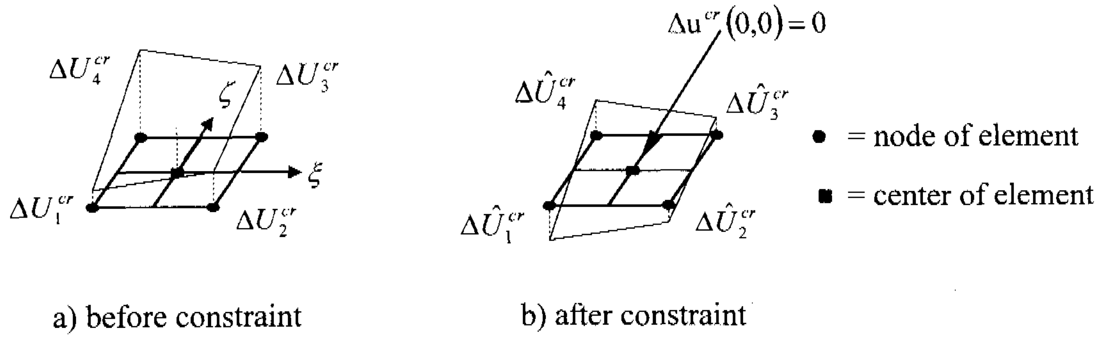

Fig. 4.2 Schematic illustration of constraint to remove rigid-body crack displacement in the x-direction

To constrain the rigid-body crack displacement increment in the x-direction, we select to set the crack displacement increment in the x-direction, at the center of the element, to zero, i.e.,

$$\Delta u^{cr}(0,0)=0=\frac{\Delta U_1^{cr}}{4}+\frac{\Delta U_2^{cr}}{4}+\frac{\Delta U_3^{cr}}{4}+\frac{\Delta U_4^{cr}}{4} \tag{4.29}$$

which gives the constraint equation to remove the rigid-body crack displacement increment in the x-direction as

$$\Delta U_1^{cr}+\Delta U_2^{cr}+\Delta U_3^{cr}+\Delta U_4^{cr}=0. \tag{4.30}$$

For the rigid-body crack displacement increment in the y-direction, the same idea is applied, i.e.,

$$\Delta v^{cr}(0,0)=0=\frac{\Delta V_1^{cr}}{4}+\frac{\Delta V_2^{cr}}{4}+\frac{\Delta V_3^{cr}}{4}+\frac{\Delta V_4^{cr}}{4} \tag{4.31}$$

which gives the constraint equation to remove the rigid-body crack displacement in the y-direction as

$$\Delta V_1^{cr} + \Delta V_2^{cr} + \Delta V_3^{cr} + \Delta V_4^{cr} = 0 . \tag{4.32}$$

To constrain the rigid-body crack rotation increment, we propose, without the loss of generality, to set $\frac{\partial\left(\Delta v^{cr}(0,0)\right)}{\partial x} = 0$. The condition leads to

$$\frac{\partial\left(\Delta v^{cr}(0,0)\right)}{\partial x} = 0 = -\frac{\Delta_1 V^{cr}}{4} + \frac{\Delta_2 V^{cr}}{4} + \frac{\Delta_3 V^{cr}}{4} - \frac{\Delta_4 V^{cr}}{4} \tag{4.33}$$

which gives the constraint equation to remove rigid-body crack rotation increment as

$$-\Delta V_1^{cr} + \Delta V_2^{cr} - \Delta V_3^{cr} + \Delta V_4^{cr} = 0 . \tag{4.34}$$

Applying the constraint equations to Eq. (4.23), we get

$$\hat{\mathbf{K}}^{cr} \Delta \hat{\mathbf{U}}^{cr} = \Delta \hat{\mathbf{R}}^{cr} \tag{4.35}$$

The stability condition can be obtained by checking the eigenvalues of $\hat{\mathbf{K}}^{cr}$. If all the eigenvalues are positive, it means the equilibrium is stable and there is no bifurcation.