CHAPTER 5

RESULTS AND DISCUSSION

The cracking localization using a modified smeared crack method is proposed in this study. The derivation of the method is shown in Chapter 4. To show its applicability, the proposed method is tried with 3 problems of cracking localization as

5.1 Uniaxial Problem using One-Dimensional Bar Elements.
5.2 Uniaxial Problem using Two-Dimensional Four-Noded Elements
5.3 Uniaxial Problem using Various Number of Four-Noded Elements.

5.1 Uniaxial Problem using One-Dimensional Bar Elements

Consider a one-dimensional bar in Fig. 5.1. The bar has one fixed support at one end. At the other end, controlled displacement $\bar{u}$ is applied. The length of the bar is $2L$ and the area is A. The material is assumed to be elastic with Young's modulus equal to E. The bar is discretized into two elements, each of which has the length of L. Each element can accommodate one crack. The characteristic length of each crack, in this case, is equal to the length of the element. The conventional linear shape function is used for the displacement interpolation.

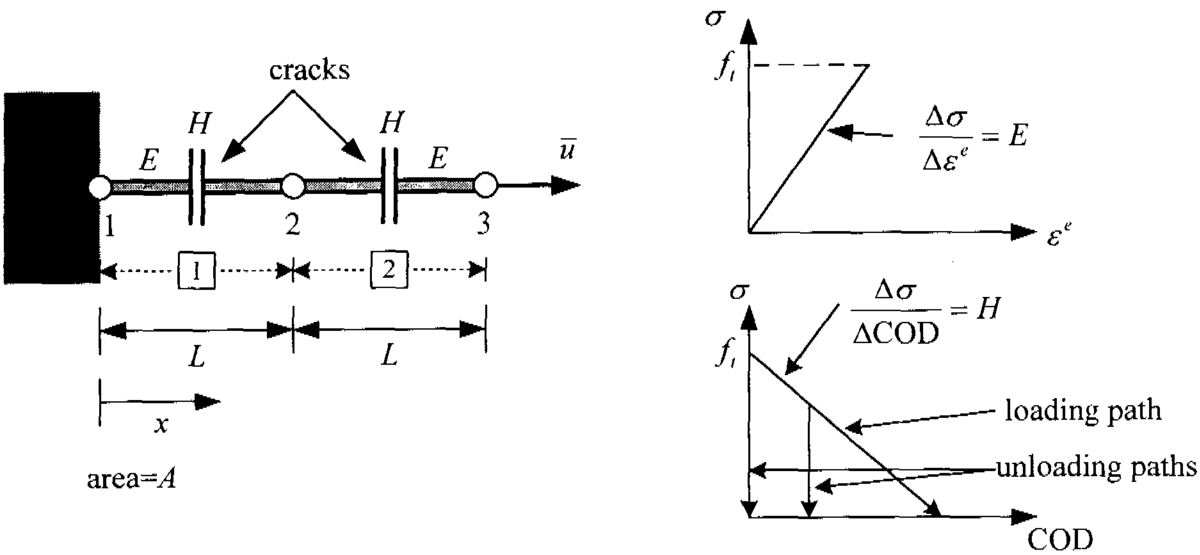

Fig. 5.1 Uniaxial problem using two 1-D bar elements

Assume no crack at the beginning of loading. After that, the controlled displacement is increased until the stress of the bar reaches the tensile strength f_t. By the strength criterion, both elements are cracked. The cracks follow the constitutive law for cracks. For opening cracks, a linear relationship between the transmitted tensile stress and the crack opening displacement (COD) with the slope $\frac{\Delta\sigma}{\Delta \mathrm{COD}}$ equal to H is assumed. For each of unloading cracks, a vertical unloading path with constant COD equal to the existing COD is applied (see Fig. 5.1).

Consider an incremental step after the initiation of the cracks. The stiffness equation for the cracked elements 1 and 2 is written as

$$\frac{A}{L}\begin{bmatrix} E & -E & -E & E \\ -E & E & E & -E \\ -E & E & E+\tilde{H} & -(E+\tilde{H}) \\ E & -E & -(E+\tilde{H}) & E+\tilde{H} \end{bmatrix}\begin{bmatrix} \Delta U_j \\ \Delta U_k \\ \Delta^i U_j^{cr} \\ \Delta^i U_k^{cr} \end{bmatrix} = \begin{bmatrix} \Delta F_j^e \\ \Delta F_k^e \\ 0 \\ 0 \end{bmatrix} \tag{5.2}$$

where $\tilde{H} = \frac{H}{L}$. Here, $i=1, j=1, k=2$ for element 1 and $i=2, j=2, k=3$ for element 2, and i denotes the element number while j and k denote the node numbers.

Assembling all the element stiffness equations, we get

$$\frac{A}{L}\begin{bmatrix} E & -E & 0 & -E & E & 0 & 0 \\ -E & 2E & -E & E & -E & -E & E \\ 0 & -E & E & 0 & 0 & E & -E \\ -E & E & 0 & E+\tilde{H} & -(E+\tilde{H}) & 0 & 0 \\ E & -E & 0 & -(E+\tilde{H}) & E+\tilde{H} & 0 & 0 \\ 0 & -E & E & 0 & 0 & E+\tilde{H} & -(E+\tilde{H}) \\ 0 & E & -E & 0 & 0 & -(E+\tilde{H}) & E+\tilde{H} \end{bmatrix}\begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ \Delta U_3 \\ \Delta^1 U_1^{cr} \\ \Delta^1 U_2^{cr} \\ \Delta^2 U_2^{cr} \\ \Delta^2 U_3^{cr} \end{Bmatrix} = \begin{Bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{5.3}$$

Applying the prescribed conditions, we obtain

$$\frac{A}{L}\begin{bmatrix} 2E & E & -E & -E & E \\ E & E+\tilde{H} & -(E+\tilde{H}) & 0 & 0 \\ -E & -(E+\tilde{H}) & E+\tilde{H} & 0 & 0 \\ -E & 0 & 0 & E+\tilde{H} & -(E+\tilde{H}) \\ E & 0 & 0 & -(E+\tilde{H}) & E+\tilde{H} \end{bmatrix}\begin{Bmatrix} \Delta U_2 \\ \Delta^1 U_1^{cr} \\ \Delta^1 U_2^{cr} \\ \Delta^2 U_2^{cr} \\ \Delta^2 U_3^{cr} \end{Bmatrix} = \frac{A}{L}\begin{Bmatrix} E\Delta\bar{u} \\ 0 \\ 0 \\ -E\Delta\bar{u} \\ E\Delta\bar{u} \end{Bmatrix}. \tag{5.4}$$

Removing the total displacement increment degree of freedom ΔU_2 by the static condensation (see Appendix A) gives

$$\frac{A}{L}\begin{bmatrix} \frac{E+2\widetilde{H}}{2} & -\frac{E+2\widetilde{H}}{2} & \frac{E}{2} & -\frac{E}{2} \\ -\frac{E+2\widetilde{H}}{2} & \frac{E+2\widetilde{H}}{2} & -\frac{E}{2} & \frac{E}{2} \\ \frac{E}{2} & -\frac{E}{2} & \frac{E+2\widetilde{H}}{2} & -\frac{E+2\widetilde{H}}{2} \\ -\frac{E}{2} & \frac{E}{2} & -\frac{E+2\widetilde{H}}{2} & \frac{E+2\widetilde{H}}{2} \end{bmatrix} \begin{Bmatrix} \Delta^1 U_1^{cr} \\ \Delta^1 U_2^{cr} \\ \Delta^2 U_2^{cr} \\ \Delta^2 U_3^{cr} \end{Bmatrix} = \frac{A}{L}\begin{Bmatrix} -\frac{E\Delta\overline{u}}{2} \\ \frac{E\Delta\overline{u}}{2} \\ -\frac{E\Delta\overline{u}}{2} \\ \frac{E\Delta\overline{u}}{2} \end{Bmatrix}. \quad (5.5)$$

It should be noted that the above equation is singular due to the rigid-body crack displacement increments in the two elements. For one-dimensional problems, the crack displacement increment at the center of each element is set to zero [Eq. (4.27)], i.e.,

$$\Delta^1 U_1^{cr} + \Delta^1 U_2^{cr} = 0 \text{ and } \Delta^2 U_2^{cr} + \Delta^2 U_3^{cr} = 0. \quad (5.6)$$

Using Eq. (5.6) in Eq. (5.5) leads to

$$\frac{A}{L}\begin{bmatrix} (E+2\widetilde{H}) & E \\ E & (E+2\widetilde{H}) \end{bmatrix} \begin{Bmatrix} \Delta^1 U_1^{cr} \\ \Delta^2 U_1^{cr} \end{Bmatrix} = \frac{A}{2L}\begin{Bmatrix} -E\Delta\overline{u} \\ -E\Delta\overline{u} \end{Bmatrix}. \quad (5.7)$$

Note that, in applying the constraints to Eq. (5.5), not only the row but also the column operations must be performed to the stiffness matrix so as to obtain the symmetric matrix in Eq. (5.7). Moreover, the constraints may be earlier applied to each element before assembling the element stiffness equations.

The eigenvalues of the obtained stiffness matrix are $\frac{2A\widetilde{H}}{L}$ and $\frac{2A(E+\widetilde{H})}{L}$. Both eigenvalues are positive only when $\widetilde{H}>0$. This means that the crack pattern having two cracks opening at the same time is unstable unless hardening behavior occurs at the cracks.

If we assume that the crack in the element 2 undergoes the elastic unloading, the stiffness equation becomes

$$\frac{A}{L}\begin{bmatrix} E & -E & 0 & -E & E \\ -E & 2E & -E & E & -E \\ 0 & -E & E & 0 & 0 \\ -E & E & 0 & E+\widetilde{H} & -(E+\widetilde{H}) \\ E & -E & 0 & -(E+\widetilde{H}) & E+\widetilde{H} \end{bmatrix} \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ \Delta U_3 \\ \Delta^1 U_1^{cr} \\ \Delta^1 U_2^{cr} \end{Bmatrix} = \begin{Bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \\ 0 \\ 0 \end{Bmatrix}. \quad (5.8)$$

Employing the same process (see Appendix A), we finally obtain

$$\frac{A}{L}\left[(E+2\widetilde{H})\right]\left\{\Delta^1 U_1^{cr}\right\} = \frac{A}{2L}\left\{-E\Delta\overline{u}\right\}. \quad (5.9)$$

The eigenvalue of the stiffness matrix is $\dfrac{A(E+2\widetilde{H})}{L}$ which is positive when $\widetilde{H} > -\dfrac{E}{2}$. Assuming that the crack in the element 1 undergoes the elastic unloading will yield the same conclusion.

In summary, immediately after the two elements are cracked due to the strength criterion employed, the equilibrium path is unstable and bifurcation occurs unless the two cracks exhibit hardening behavior, i.e., when $\widetilde{H}>0$. In reality, cracks will exhibit softening behavior. Therefore, the two cracks cannot continue to open at the same time. If one of the cracks undergoes the elastic unloading, the stable equilibrium path can be observed as long as $\widetilde{H} > -\dfrac{E}{2}$. As shown in Fig. 5.2, the cases where $\widetilde{H} < -\dfrac{E}{2}$ represent the responses with snapback. Under the displacement control, the snapback responses are always unstable.

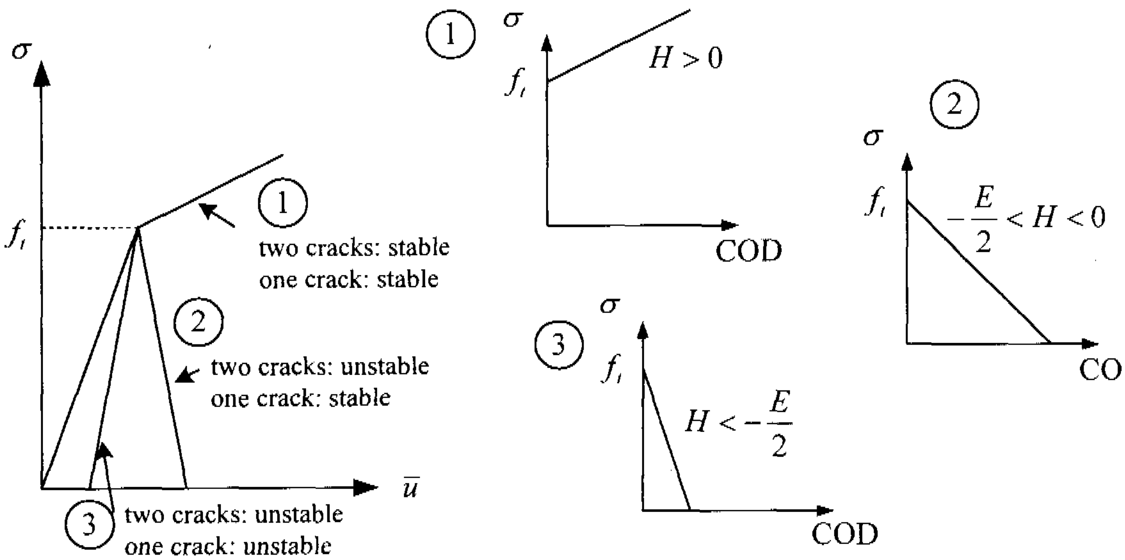

Fig. 5.2 Responses of the uniaxial problem using two 1-D bar elements

5.2 Uniaxial Problem using Two-Dimensional Four-Noded Elements

Consider Fig. 5.3. It is an axial bar with a fixed support at one end and controlled displacement at the other. The total length of the bar is 4, the thickness of the bar is 1, and the cross-sectional area is 2. It is discretized into two four-noded quadrilateral elements as shown in the same figure. The material is assumed to be elastic with Young's Modulus, Poisson's ratio, and tensile strength equal to E, 0, and f_t, respectively. When an element is cracked, the constitutive matrix of the cracked element is

$$\hat{\mathbf{D}}_{cr} = \begin{bmatrix} \widetilde{H} & 0 \\ 0 & \widetilde{G} \end{bmatrix}$$

where $\widetilde{H} = \dfrac{H}{L}$, and H is defined in the previous problem. For analytical analysis, $\widetilde{G}$ is assumed to be zero.

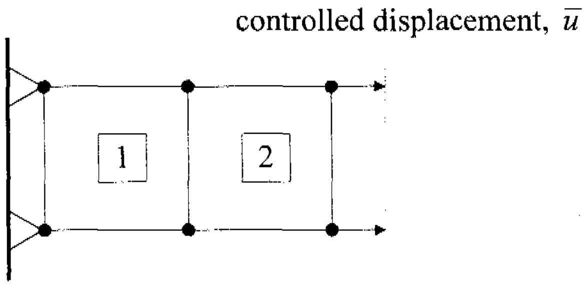

Fig. 5.3 Uniaxial problem using two 2-D four-noded elements

The controlled displacement is increased until the tensile stress in all elements reaches the tensile strength. Under the strength criterion, all elements are then cracked. At this state of the equilibrium, Eq. (4.35) is formed and the eigenvalues of the stiffness $\hat{\mathbf{K}}_{cr}$ in Eq. (4.35) are computed analytically (see Appendix B) as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{4}{17}\left(-6+\sqrt{2}\right)E \approx 1.08E > 0 \\ \frac{4}{17}\left(6+\sqrt{2}\right)E \approx 1.75E > 0 \\ -\frac{1}{3}\left(-4+\sqrt{7}\right)\widetilde{H} \approx 0.45\widetilde{H} < 0 \\ \frac{1}{3}\left(4+\sqrt{7}\right)\widetilde{H} \approx 2.22\widetilde{H} < 0 \\ \frac{1}{6}\left(9E+8\widetilde{H}-\sqrt{27E^2+54E\widetilde{H}+28\widetilde{H}^2}\right) > 0 \text{ if } \widetilde{H} > -E \\ \frac{1}{6}\left(9E+8\widetilde{H}-\sqrt{27E^2+54E\widetilde{H}+28\widetilde{H}^2}\right) > 0 \text{ if } \widetilde{H} > -\frac{3E}{2} \end{bmatrix}$$

Four zero eigenvalues appear because the mode II crack stiffness modulus $\widetilde{G}$ is assumed to be 0. Since the softening behavior is always expected in quasi-brittle materials, $\widetilde{H}$ is assumed to be negative. Therefore, the eigenvalues of $0.45\widetilde{H}$ and

$2.22\tilde{H}$ are always negative. Consequently, we may conclude that the equilibrium state is unstable, and the bifurcation occurs in this case.

The method is also tried numerically. We use the finite element analysis program called Feap++ (Nanakorn 1996) to construct the stiffness equation numerically. For the material properties, we assume that both Young's Modulus and tensile strength are equal to 1. Furthermore, H is assumed to be equal to $-\frac{1}{6}$. The tension-softening curve used is shown in Fig 5.4. The characteristic length is equal to the width of the element. Further assume $\tilde{G}$ equal to 0.001 to avoid zero eigenvalues due to spurious shear crack modes.

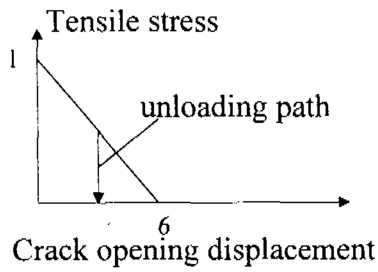

Fig. 5.4 Tension-softening curve for problem 5.2

By performing the eigenvalue analysis, we get the eigenvalues as

$$\begin{bmatrix} 0.0001 \\ 0.0005 \\ 0.001 \\ 0.001 \\ 1.08 \\ 1.75 \\ -0.08 \\ -0.37 \\ 0.56 \\ 2 \end{bmatrix}.$$

The eigenvalues obtained show that in this case if the two cracks open simultaneously, the equilibrium state will be unstable. The bifurcation occurs in this case. One of the elements will have to undergo the elastic unloading to make the equilibrium stable. The element that undergoes the elastic unloading is incrementally considered as an elastic element without crack. In this problem, there are two alternatives. By the eigenvalue analysis, we obtain the following eigenvalues for the two cases:

crack in the first elements opens   crack in the second element opens

$$\begin{bmatrix} 0.001 \\ 0.24 \\ 0.82 \\ 0.64 \\ 1.48 \end{bmatrix} \qquad \begin{bmatrix} 0.001 \\ 0.24 \\ 0.82 \\ 0.0003 \\ 0.71 \end{bmatrix}$$

From the eigenvalues, it is found that if only a crack in one element is assumed to be opened, all the eigenvalues will be positive. The equilibrium state will be stable.

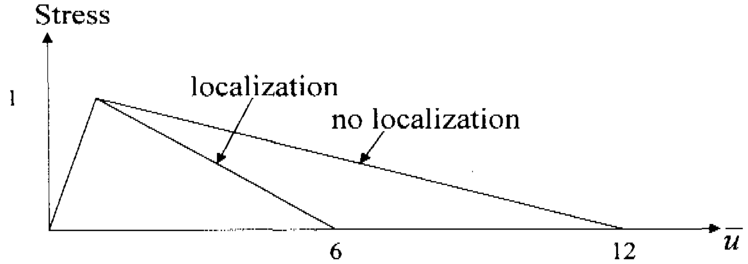

Fig. 5.5 Comparison of responses between the cases with and without localization consideration

Fig. 5.5 shows responses in comparison of the cases with and without localization consideration. The responses and the dissipated energy of these two cases are different. It is noted, from this problem, that the consideration of localization is necessary for fracture problems.

5.3 Uniaxial Problem using Various Number of Four-Noded Elements

In order to show the validity of the method in real and more complicated problems, a cracking localization problem shown in Fig. 5.6 is considered. It is an axial bar with a fixed support. At the other end, the displacement is controlled. The total length of the bar is 100, and the cross-sectional area is 25×50. The bar is discretized into two, five, and ten four-noded quadrilateral elements as shown in the same figure.

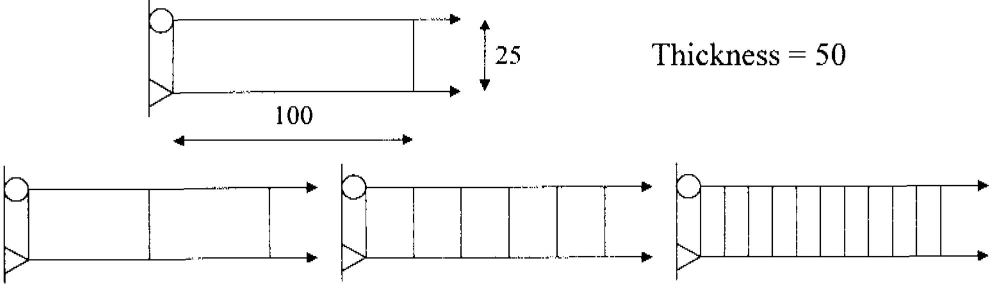

Fig. 5.6 Uniaxial problem using various number of four-noded elements

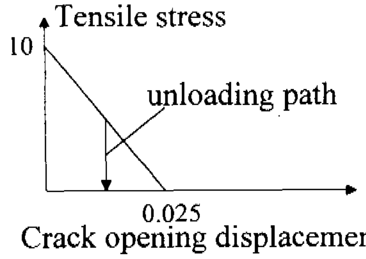

Fig. 5.7 Tension-softening curve for problem 5.3

The material is assumed to be elastic with Young's modulus, Poisson's ratio, and the tensile strength equal to 100000, 0, and 10, respectively. The tension-softening curve used in the problem is shown in Fig. 5.7. The characteristic length will be equal to the width of the element.

The controlled displacement is increased until the tensile stress in all elements reaches the tensile strength. Under the strength criterion, all elements are then cracked. At this state of the equilibrium, Eq. (4.35) is formed and the eigenvalues of the stiffness matrix $\hat{\mathbf{K}}_{cr}$ in Eq. (4.35) are computed. It is found that, for all meshes with two, five, and ten elements, some of the eigenvalues are negative; therefore, the equilibrium state is unstable and there is bifurcation. This means that some elements may undergo the elastic unloading. Those elements that undergo the elastic unloading are incrementally considered as elastic elements without cracks. To figure out which elements will have opening cracks and which elements will exhibit the elastic unloading, we consider all possible combinations. The stiffness $\hat{\mathbf{K}}_{cr}$'s for all different cases are formed and the eigenvalues are calculated. From the result, it is found that only cases, where there is only one opening crack in one of the two elements, result in all positive eigenvalues. It can be interpreted that the stable solutions are the solutions that have only one crack. Nevertheless, the location of the only one crack can be in any element.

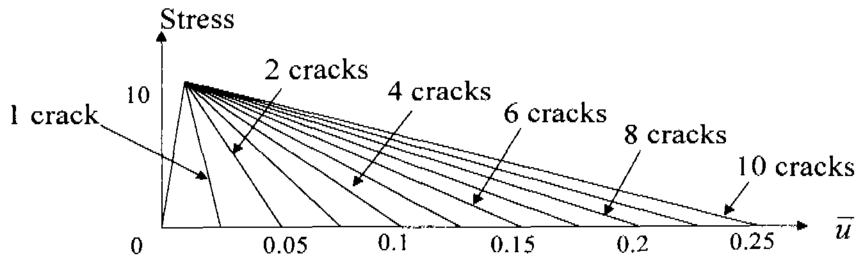

Fig. 5.8 Responses for different numbers of cracks

Fig 5.8 shows responses for different numbers of cracks. The responses with one crack are the only stable responses. The results from the meshes with two, five, and ten elements are exactly the same for the same number of the opening cracks. This is because of the proper use of the characteristic length.