

APPENDIX B

Calculation for Problem 5.2 by the Mathematica Program

(*****
 (*This is the program for problem 5.2*)
 (*****)

(*This is to construct the uncracked and cracked stiffness as Ke and Kc1*)
 v = 0

$$De = \frac{E}{1-\nu^2} \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & \nu \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$$

$$Dcr1 = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}$$

$$n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Dc1 = n \cdot Dcr1 \cdot \text{Transpose}[n]$$

$$B = \frac{1}{4} \begin{pmatrix} -(1-y) & 0 & 1-y & 0 & 1+y & 0 & -(1+y) & 0 \\ 0 & -(1-x) & 0 & -(1+x) & 0 & 1+x & 0 & 1-x \\ -(1-x) & -(1-y) & -(1+x) & 1-y & 1+x & 1+y & 1-x & -(1+y) \end{pmatrix}$$

$$\text{Temp1} = \text{Transpose}[B] \cdot De \cdot B$$

$$\text{Temp2} = \text{Transpose}[B] \cdot Dc1 \cdot B$$

$$Ke = \int_{-1}^1 \int_{-1}^1 \text{Temp1} \, dx \, dy$$

$$Kc1 = \int_{-1}^1 \int_{-1}^1 \text{Temp2} \, dx \, dy$$

0

$$\{(E, 0, 0), (0, E, 0), (0, 0, \frac{E}{2})\}$$

$$\{(H, 0), (0, 0)\}$$

$$\{(1, 0), (0, 0), (0, 1)\}$$

$$\{(H, 0, 0), (0, 0, 0), (0, 0, 0)\}$$

$$\left\{ \left\{ \frac{1}{4}(-1+y), 0, \frac{1-y}{4}, 0, \frac{1+y}{4}, 0, \frac{1}{4}(-1-y), 0 \right\}, \left\{ 0, \frac{1}{4}(-1+x), 0, \frac{1}{4}(-1-x), 0, \frac{1+x}{4}, 0, \frac{1-x}{4} \right\}, \right. \\ \left. \left\{ \frac{1}{4}(-1+x), \frac{1}{4}(-1+y), \frac{1}{4}(-1-x), \frac{1-y}{4}, \frac{1+x}{4}, \frac{1+y}{4}, \frac{1-x}{4}, \frac{1}{4}(-1-y) \right\} \right\}$$

$$\begin{aligned}
& \left\{ \frac{1}{32} E(-1+X)^2 + \frac{1}{16} E(-1+Y)^2, \frac{1}{32} E(-1+X)(-1+Y), \frac{1}{32} E(-1-X)(-1+X) + \frac{1}{16} E(1-Y)(-1+Y), \right. \\
& \quad \frac{1}{32} E(-1+X)(1-Y), \frac{1}{32} E(-1+X)(1+X) + \frac{1}{16} E(-1+Y)(1+Y), \frac{1}{32} E(-1+X)(1+Y), \\
& \quad \left. \frac{1}{32} E(1-X)(-1+X) + \frac{1}{16} E(-1-Y)(-1+Y), \frac{1}{32} E(-1+X)(-1-Y) \right\}, \left\{ \frac{1}{32} E(-1+X)(-1+Y), \right. \\
& \quad \frac{1}{16} E(-1+X)^2 + \frac{1}{32} E(-1+Y)^2, \frac{1}{32} E(-1-X)(-1+Y), \frac{1}{16} E(-1-X)(-1+X) + \frac{1}{32} E(1-Y)(-1+Y), \\
& \quad \frac{1}{32} E(1+X)(-1+Y), \frac{1}{16} E(-1+X)(1+X) + \frac{1}{32} E(-1+Y)(1+Y), \\
& \quad \left. \frac{1}{32} E(1-X)(-1+Y), \frac{1}{16} E(1-X)(-1+X) + \frac{1}{32} E(-1-Y)(-1+Y) \right\}, \\
& \left\{ \frac{1}{32} E(-1-X)(-1+X) + \frac{1}{16} E(1-Y)(-1+Y), \frac{1}{32} E(-1-X)(-1+Y), \right. \\
& \quad \frac{1}{32} E(-1-X)^2 + \frac{1}{16} E(1-Y)^2, \frac{1}{32} E(-1-X)(1-Y), \frac{1}{32} E(-1-X)(1+X) + \frac{1}{16} E(1-Y)(1+Y), \\
& \quad \frac{1}{32} E(-1-X)(1+Y), \frac{1}{32} E(-1-X)(1-X) + \frac{1}{16} E(-1-Y)(1-Y), \frac{1}{32} E(-1-X)(-1-Y) \left. \right\}, \\
& \left\{ \frac{1}{32} E(-1+X)(1-Y), \frac{1}{16} E(-1-X)(-1+X) + \frac{1}{32} E(1-Y)(-1+Y), \frac{1}{32} E(-1-X)(1-Y), \right. \\
& \quad \frac{1}{16} E(-1-X)^2 + \frac{1}{32} E(1-Y)^2, \frac{1}{32} E(1+X)(1-Y), \frac{1}{16} E(-1-X)(1+X) + \frac{1}{32} E(1-Y)(1+Y), \\
& \quad \frac{1}{32} E(1-X)(1-Y), \frac{1}{16} E(-1-X)(1-X) + \frac{1}{32} E(-1-Y)(1-Y) \left. \right\}, \\
& \left\{ \frac{1}{32} E(-1+X)(1+X) + \frac{1}{16} E(-1+Y)(1+Y), \frac{1}{32} E(1+X)(-1+Y), \right. \\
& \quad \frac{1}{32} E(-1-X)(1+X) + \frac{1}{16} E(1-Y)(1+Y), \frac{1}{32} E(1+X)(1-Y), \frac{1}{32} E(1+X)^2 + \frac{1}{16} E(1+Y)^2, \\
& \quad \frac{1}{32} E(1+X)(1+Y), \frac{1}{32} E(1-X)(1+X) + \frac{1}{16} E(-1-Y)(1+Y), \frac{1}{32} E(1+X)(-1-Y) \left. \right\}, \\
& \left\{ \frac{1}{32} E(-1+X)(1+Y), \frac{1}{16} E(-1+X)(1+X) + \frac{1}{32} E(-1+Y)(1+Y), \right. \\
& \quad \frac{1}{32} E(-1-X)(1+Y), \frac{1}{16} E(-1-X)(1+X) + \frac{1}{32} E(1-Y)(1+Y), \frac{1}{32} E(1+X)(1+Y), \\
& \quad \frac{1}{16} E(1+X)^2 + \frac{1}{32} E(1+Y)^2, \frac{1}{32} E(1-X)(1+Y), \frac{1}{16} E(1-X)(1+X) + \frac{1}{32} E(-1-Y)(1+Y) \left. \right\}, \\
& \left\{ \frac{1}{32} E(1-X)(-1+X) + \frac{1}{16} E(-1-Y)(-1+Y), \right. \\
& \quad \frac{1}{32} E(1-X)(-1+Y), \frac{1}{32} E(-1-X)(1-X) + \frac{1}{16} E(-1-Y)(1-Y), \\
& \quad \frac{1}{32} E(1-X)(1-Y), \frac{1}{32} E(1-X)(1+X) + \frac{1}{16} E(-1-Y)(1+Y), \\
& \quad \frac{1}{32} E(1-X)(1+Y), \frac{1}{32} E(1-X)^2 + \frac{1}{16} E(-1-Y)^2, \frac{1}{32} E(1-X)(-1-Y) \left. \right\}, \\
& \left\{ \frac{1}{32} E(-1+X)(-1-Y), \frac{1}{16} E(1-X)(-1+X) + \frac{1}{32} E(-1-Y)(-1+Y), \right. \\
& \quad \frac{1}{32} E(-1-X)(-1-Y), \frac{1}{16} E(-1-X)(1-X) + \frac{1}{32} E(-1-Y)(1-Y), \frac{1}{32} E(1+X)(-1-Y), \\
& \quad \frac{1}{16} E(1-X)(1+X) + \frac{1}{32} E(-1-Y)(1+Y), \frac{1}{32} E(1-X)(-1-Y), \frac{1}{16} E(1-X)^2 + \frac{1}{32} E(-1-Y)^2 \left. \right\}
\end{aligned}$$

$$\begin{aligned} & \left\{ \left\{ \frac{1}{16} H(-1+Y)^2, 0, \frac{1}{16} H(1-Y)(-1+Y), 0, \frac{1}{16} H(-1+Y)(1+Y), 0, \frac{1}{16} H(-1-Y)(-1+Y), 0 \right\}, \right. \\ & (0, 0, 0, 0, 0, 0, 0, 0), \left\{ \frac{1}{16} H(1-Y)(-1+Y), 0, \frac{1}{16} H(1-Y)^2, 0, \frac{1}{16} H(1-Y)(1+Y), \right. \\ & \left. 0, \frac{1}{16} H(-1-Y)(1-Y), 0 \right\}, (0, 0, 0, 0, 0, 0, 0, 0), \left\{ \frac{1}{16} H(-1+Y)(1+Y), 0, \right. \\ & \left. \frac{1}{16} H(1-Y)(1+Y), 0, \frac{1}{16} H(1+Y)^2, 0, \frac{1}{16} H(-1-Y)(1+Y), 0 \right\}, (0, 0, 0, 0, 0, 0, 0, 0), \\ & \left. \left\{ \frac{1}{16} H(-1-Y)(-1+Y), 0, \frac{1}{16} H(-1-Y)(1-Y), 0, \frac{1}{16} H(-1-Y)(1+Y), 0, \frac{1}{16} H(-1-Y)^2, 0 \right\}, \right. \\ & \left. (0, 0, 0, 0, 0, 0, 0, 0) \right\} \end{aligned}$$

$$\begin{aligned} & \left\{ \left\{ \frac{E}{2}, \frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, 0, \frac{E}{8} \right\}, \left\{ \frac{E}{8}, \frac{E}{2}, \frac{E}{8}, 0, -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, -\frac{E}{4} \right\}, \right. \\ & \left\{ -\frac{E}{4}, \frac{E}{8}, \frac{E}{2}, -\frac{E}{8}, 0, -\frac{E}{8}, -\frac{E}{4}, \frac{E}{8} \right\}, \left\{ -\frac{E}{8}, 0, -\frac{E}{8}, \frac{E}{2}, \frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, -\frac{E}{4} \right\}, \\ & \left\{ -\frac{E}{4}, -\frac{E}{8}, 0, \frac{E}{8}, \frac{E}{2}, \frac{E}{8}, -\frac{E}{4}, -\frac{E}{8} \right\}, \left\{ -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, \frac{E}{2}, \frac{E}{8}, 0 \right\}, \\ & \left. \left\{ 0, -\frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, \frac{E}{2}, -\frac{E}{8} \right\}, \left\{ \frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, 0, -\frac{E}{8}, \frac{E}{2} \right\} \right\} \end{aligned}$$

$$\begin{aligned} & \left\{ \left\{ \frac{H}{3}, 0, -\frac{H}{3}, 0, -\frac{H}{6}, 0, \frac{H}{6}, 0 \right\}, (0, 0, 0, 0, 0, 0, 0, 0), \left\{ -\frac{H}{3}, 0, \frac{H}{3}, 0, \frac{H}{6}, 0, -\frac{H}{6}, 0 \right\}, \right. \\ & (0, 0, 0, 0, 0, 0, 0, 0), \left\{ -\frac{H}{6}, 0, \frac{H}{6}, 0, \frac{H}{3}, 0, -\frac{H}{3}, 0 \right\}, (0, 0, 0, 0, 0, 0, 0, 0), \\ & \left. \left\{ \frac{H}{6}, 0, -\frac{H}{6}, 0, -\frac{H}{3}, 0, \frac{H}{3}, 0 \right\}, (0, 0, 0, 0, 0, 0, 0, 0) \right\} \end{aligned}$$

(*This is to construct the matrix before static condensation*)

Ka =

{Ke[[3, 3]], Ke[[3, 4]], 0, Ke[[3, 5]], Ke[[3, 6]], 0},
 {Ke[[4, 3]], Ke[[4, 4]], 0, Ke[[4, 5]], Ke[[4, 6]], 0},
 {0, 0, 0, 0, 0, 0},
 {Ke[[5, 3]], Ke[[5, 4]], 0, Ke[[5, 5]], Ke[[5, 6]], 0},
 {Ke[[6, 3]], Ke[[6, 4]], 0, Ke[[6, 5]], Ke[[6, 6]], 0},
 {0, 0, 0, 0, 0, 0}

Kb =

{Ke[[1, 1]], Ke[[1, 2]], Ke[[1, 4]], Ke[[1, 7]], Ke[[1, 8]], Ke[[1, 6]]},
 {Ke[[2, 1]], Ke[[2, 2]], Ke[[2, 4]], Ke[[2, 7]], Ke[[2, 8]], Ke[[2, 6]]},
 {Ke[[4, 1]], Ke[[4, 2]], Ke[[4, 4]], Ke[[4, 7]], Ke[[4, 8]], Ke[[4, 6]]},
 {Ke[[7, 1]], Ke[[7, 2]], Ke[[7, 4]], Ke[[7, 7]], Ke[[7, 8]], Ke[[7, 6]]},
 {Ke[[8, 1]], Ke[[8, 2]], Ke[[8, 4]], Ke[[8, 7]], Ke[[8, 8]], Ke[[8, 6]]},
 {Ke[[6, 1]], Ke[[6, 2]], Ke[[6, 4]], Ke[[6, 7]], Ke[[6, 8]], Ke[[6, 6]]}

K11 = Ka + Kb

K12 = -

{Ke[[3, 1]], Ke[[3, 2]], Ke[[3, 3]], Ke[[3, 4]], Ke[[3, 5]], Ke[[3, 6]], Ke[[3, 7]], Ke[[3, 8]],
 Ke[[1, 1]], Ke[[1, 2]], Ke[[1, 3]], Ke[[1, 4]], Ke[[1, 5]], Ke[[1, 6]], Ke[[1, 7]], Ke[[1, 8]]},
 {Ke[[4, 1]], Ke[[4, 2]], Ke[[4, 3]], Ke[[4, 4]], Ke[[4, 5]], Ke[[4, 6]], Ke[[4, 7]], Ke[[4, 8]],
 Ke[[2, 1]], Ke[[2, 2]], Ke[[2, 3]], Ke[[2, 4]], Ke[[2, 5]], Ke[[2, 6]], Ke[[2, 7]], Ke[[2, 8]]},
 {0, 0, 0, 0, 0, 0, 0, 0},
 {Ke[[4, 1]], Ke[[4, 2]], Ke[[4, 3]], Ke[[4, 4]], Ke[[4, 5]], Ke[[4, 6]], Ke[[4, 7]], Ke[[4, 8]]},
 {Ke[[5, 1]], Ke[[5, 2]], Ke[[5, 3]], Ke[[5, 4]], Ke[[5, 5]], Ke[[5, 6]], Ke[[5, 7]], Ke[[5, 8]],
 Ke[[7, 1]], Ke[[7, 2]], Ke[[7, 3]], Ke[[7, 4]], Ke[[7, 5]], Ke[[7, 6]], Ke[[7, 7]], Ke[[7, 8]]},
 {Ke[[6, 1]], Ke[[6, 2]], Ke[[6, 3]], Ke[[6, 4]], Ke[[6, 5]], Ke[[6, 6]], Ke[[6, 7]], Ke[[6, 8]]},
 {Ke[[8, 1]], Ke[[8, 2]], Ke[[8, 3]], Ke[[8, 4]], Ke[[8, 5]], Ke[[8, 6]], Ke[[8, 7]], Ke[[8, 8]]},
 {0, 0, 0, 0, 0, 0, 0, 0},
 {Ke[[6, 1]], Ke[[6, 2]], Ke[[6, 3]], Ke[[6, 4]], Ke[[6, 5]], Ke[[6, 6]], Ke[[6, 7]], Ke[[6, 8]]}

K21 = Transpose[K12]

Kx = Ke + Kc1

Kz = {{0, 0, 0, 0, 0, 0, 0, 0}}

Kh = Flatten{{

Kx[[1]], **Kz**[[1]], **Kx**[[2]], **Kz**[[1]], **Kx**[[3]], **Kz**[[1]], **Kx**[[4]], **Kz**[[1]],
Kx[[5]], **Kz**[[1]], **Kx**[[6]], **Kz**[[1]], **Kx**[[7]], **Kz**[[1]], **Kx**[[8]], **Kz**[[1]],
Kz[[1]], **Kx**[[1]], **Kz**[[1]], **Kx**[[2]], **Kz**[[1]], **Kx**[[3]], **Kz**[[1]], **Kx**[[4]],
Kz[[1]], **Kx**[[5]], **Kz**[[1]], **Kx**[[6]], **Kz**[[1]], **Kx**[[7]], **Kz**[[1]], **Kx**[[8]]}}

K22 = Partition[Kh, 16]

{{ $\frac{E}{2}, -\frac{E}{8}, 0, 0, -\frac{E}{8}, 0$ }, { $-\frac{E}{8}, \frac{E}{2}, 0, \frac{E}{8}, -\frac{E}{4}, 0$ }, {0, 0, 0, 0, 0, 0}, {0, $\frac{E}{8}, 0, \frac{E}{2}, \frac{E}{8}, 0$ },
 { $-\frac{E}{8}, -\frac{E}{4}, 0, \frac{E}{8}, \frac{E}{2}, 0$ }, {0, 0, 0, 0, 0, 0}}

{{ $\frac{E}{2}, \frac{E}{8}, -\frac{E}{8}, 0, \frac{E}{8}, -\frac{E}{8}$ }, { $\frac{E}{8}, \frac{E}{2}, 0, -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{4}$ }, { $-\frac{E}{8}, 0, \frac{E}{2}, \frac{E}{8}, -\frac{E}{4}, -\frac{E}{4}$ },
 {0, $-\frac{E}{8}, \frac{E}{8}, \frac{E}{2}, -\frac{E}{8}, \frac{E}{8}$ }, { $\frac{E}{8}, -\frac{E}{4}, -\frac{E}{4}, -\frac{E}{8}, \frac{E}{2}, 0$ }, { $-\frac{E}{8}, -\frac{E}{4}, -\frac{E}{4}, \frac{E}{8}, 0, \frac{E}{2}$ }}

{{ $E, 0, -\frac{E}{8}, 0, 0, -\frac{E}{8}$ }, {0, $E, 0, 0, -\frac{E}{2}, -\frac{E}{4}$ }, { $-\frac{E}{8}, 0, \frac{E}{2}, \frac{E}{8}, -\frac{E}{4}, -\frac{E}{4}$ }, {0, 0, $\frac{E}{8}, E, 0, \frac{E}{8}$ },
 {0, $-\frac{E}{2}, -\frac{E}{4}, 0, E, 0$ }, { $-\frac{E}{8}, -\frac{E}{4}, -\frac{E}{4}, \frac{E}{8}, 0, \frac{E}{2}$ }}

$$\begin{aligned}
& \left\{ \frac{E}{2} + \frac{H}{3}, \frac{E}{8}, -\frac{E}{4} - \frac{H}{3}, -\frac{E}{8}, -\frac{E}{4} - \frac{H}{6}, -\frac{E}{8}, \frac{E}{6}, \frac{E}{8}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ \frac{E}{8}, \frac{E}{2}, \frac{E}{8}, 0, -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, -\frac{E}{4}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ -\frac{E}{4} - \frac{H}{3}, \frac{E}{8}, \frac{E}{2} + \frac{H}{3}, -\frac{E}{8}, \frac{E}{6}, -\frac{E}{8}, -\frac{E}{4} - \frac{H}{6}, \frac{E}{8}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ -\frac{E}{8}, 0, -\frac{E}{8}, \frac{E}{2}, \frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, -\frac{E}{4}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ -\frac{E}{4} - \frac{H}{6}, -\frac{E}{8}, \frac{E}{6}, \frac{E}{8}, \frac{E}{2} + \frac{H}{3}, \frac{E}{8}, -\frac{E}{4} - \frac{H}{3}, -\frac{E}{8}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, \frac{E}{2}, \frac{E}{8}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ \frac{H}{6}, -\frac{E}{8}, -\frac{E}{4} - \frac{H}{6}, \frac{E}{8}, -\frac{E}{4} - \frac{H}{3}, \frac{E}{8}, \frac{E}{2} + \frac{H}{3}, -\frac{E}{8}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ \frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, 0, -\frac{E}{8}, \frac{E}{2}, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, \frac{E}{2} + \frac{H}{3}, \frac{E}{8}, -\frac{E}{4} - \frac{H}{3}, -\frac{E}{8}, -\frac{E}{4} - \frac{H}{6}, -\frac{E}{8}, \frac{E}{6}, \frac{E}{8} \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, \frac{E}{8}, \frac{E}{2}, \frac{E}{8}, 0, -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, -\frac{E}{4} \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, -\frac{E}{4} - \frac{H}{3}, \frac{E}{8}, \frac{E}{2} + \frac{H}{3}, -\frac{E}{8}, \frac{E}{6}, -\frac{E}{8}, -\frac{E}{4} - \frac{H}{6}, \frac{E}{8} \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, -\frac{E}{8}, 0, -\frac{E}{8}, \frac{E}{2}, \frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, -\frac{E}{4} \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, -\frac{E}{4} - \frac{H}{6}, -\frac{E}{8}, \frac{E}{6}, \frac{E}{8}, \frac{E}{2} + \frac{H}{3}, \frac{E}{8}, -\frac{E}{4} - \frac{H}{3}, -\frac{E}{8} \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, -\frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, \frac{E}{2}, \frac{E}{8}, 0 \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, \frac{H}{6}, -\frac{E}{8}, -\frac{E}{4} - \frac{H}{6}, \frac{E}{8}, -\frac{E}{4} - \frac{H}{3}, \frac{E}{8}, \frac{E}{2} + \frac{H}{3}, -\frac{E}{8} \right\}, \\
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, \frac{E}{8}, -\frac{E}{4}, \frac{E}{8}, -\frac{E}{4}, -\frac{E}{8}, 0, -\frac{E}{8}, \frac{E}{2} \right\}
\end{aligned}$$

(*This is to construct the matrix after static condensation*)

$$\mathbf{Kden} = \mathbf{K22} - \mathbf{K21} \cdot \mathbf{Inverse}[\mathbf{K11}] \cdot \mathbf{K12}$$

$$\left\{ \left\{ \frac{3E}{16} + \frac{H}{3}, 0, -\frac{3E}{16} - \frac{H}{3}, 0, -\frac{E}{16} - \frac{H}{6}, 0, \frac{E}{16} + \frac{H}{6}, 0, \frac{3E}{16}, 0, -\frac{3E}{16}, 0, -\frac{E}{16}, 0, \frac{E}{16}, 0 \right\}, \right. \\ \left. \left\{ 0, \frac{6E}{17}, 0, \frac{E}{17}, 0, -\frac{E}{17}, 0, -\frac{6E}{17}, 0, -\frac{E}{17}, 0, 0, 0, 0, 0, \frac{E}{17} \right\}, \right. \\ \left\{ -\frac{3E}{16} - \frac{H}{3}, 0, \frac{3E}{16} + \frac{H}{3}, 0, \frac{E}{16} + \frac{H}{6}, 0, -\frac{E}{16} - \frac{H}{6}, 0, -\frac{3E}{16}, 0, \frac{3E}{16}, 0, \frac{E}{16}, 0, -\frac{E}{16}, 0 \right\}, \\ \left\{ 0, \frac{E}{17}, 0, \frac{3E}{17}, 0, -\frac{3E}{17}, 0, -\frac{E}{17}, 0, -\frac{3E}{17}, 0, 0, 0, 0, 0, \frac{3E}{17} \right\}, \\ \left\{ -\frac{E}{16} - \frac{H}{6}, 0, \frac{E}{16} + \frac{H}{6}, 0, \frac{3E}{16} + \frac{H}{3}, 0, -\frac{3E}{16} - \frac{H}{3}, 0, -\frac{E}{16}, 0, \frac{E}{16}, 0, \frac{3E}{16}, 0, -\frac{3E}{16}, 0 \right\}, \\ \left\{ 0, -\frac{E}{17}, 0, -\frac{3E}{17}, 0, \frac{3E}{17}, 0, \frac{E}{17}, 0, \frac{3E}{17}, 0, 0, 0, 0, 0, -\frac{3E}{17} \right\}, \\ \left\{ \frac{E}{16} + \frac{H}{6}, 0, -\frac{E}{16} - \frac{H}{6}, 0, -\frac{3E}{16} - \frac{H}{3}, 0, \frac{3E}{16} + \frac{H}{3}, 0, \frac{E}{16}, 0, -\frac{E}{16}, 0, -\frac{3E}{16}, 0, \frac{3E}{16}, 0 \right\}, \\ \left\{ 0, -\frac{6E}{17}, 0, -\frac{E}{17}, 0, \frac{E}{17}, 0, \frac{6E}{17}, 0, \frac{E}{17}, 0, 0, 0, 0, 0, -\frac{E}{17} \right\}, \left\{ \frac{3E}{16}, 0, -\frac{3E}{16}, 0, -\frac{E}{16}, 0, \frac{E}{16}, \right. \\ \left. 0, \frac{3E}{16} + \frac{H}{3}, 0, -\frac{3E}{16} - \frac{H}{3}, 0, -\frac{E}{16} - \frac{H}{6}, 0, \frac{E}{16} + \frac{H}{6}, 0 \right\}, \left\{ 0, -\frac{E}{17}, 0, -\frac{E}{17}, 0, \frac{3E}{17}, \right. \\ \left. 0, \frac{E}{17}, 0, \frac{3E}{17}, 0, 0, 0, 0, 0, -\frac{3E}{17} \right\}, \left\{ -\frac{3E}{16}, 0, \frac{3E}{16}, 0, \frac{E}{16}, 0, -\frac{E}{16}, 0, -\frac{3E}{16} - \frac{H}{3}, \right. \\ \left. 0, \frac{3E}{16} + \frac{H}{3}, 0, \frac{E}{16} + \frac{H}{6}, 0, -\frac{E}{16} - \frac{H}{6}, 0 \right\}, \left\{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\ \left\{ -\frac{E}{16}, 0, \frac{E}{16}, 0, \frac{3E}{16}, 0, -\frac{3E}{16}, 0, -\frac{E}{16} - \frac{H}{6}, 0, \frac{E}{16} + \frac{H}{6}, 0, \frac{3E}{16} + \frac{H}{3}, 0, -\frac{3E}{16} - \frac{H}{3}, 0 \right\}, \\ \left\{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\ \left\{ \frac{E}{16}, 0, -\frac{E}{16}, 0, -\frac{3E}{16}, 0, \frac{3E}{16}, 0, \frac{E}{16} + \frac{H}{6}, 0, -\frac{E}{16} - \frac{H}{6}, 0, -\frac{3E}{16} - \frac{H}{3}, 0, \frac{3E}{16} + \frac{H}{3}, 0 \right\}, \\ \left\{ 0, \frac{E}{17}, 0, \frac{3E}{17}, 0, -\frac{3E}{17}, 0, -\frac{E}{17}, 0, -\frac{3E}{17}, 0, 0, 0, 0, 0, \frac{3E}{17} \right\} \left. \right\}$$

(*This is to construct the matrix after constrain*)

$$\mathbf{Kstn1} = (\mathbf{Kden}[[3]] - \mathbf{Kden}[[1]], \mathbf{Kden}[[5]] - \mathbf{Kden}[[1]], \mathbf{Kden}[[6]] - \mathbf{Kden}[[4]], \\ \mathbf{Kden}[[7]] - \mathbf{Kden}[[1]], \mathbf{Kden}[[8]] - \mathbf{Kden}[[2]], \\ \mathbf{Kden}[[11]] - \mathbf{Kden}[[9]], \mathbf{Kden}[[13]] - \mathbf{Kden}[[9]], \mathbf{Kden}[[14]] - \mathbf{Kden}[[12]], \\ \mathbf{Kden}[[15]] - \mathbf{Kden}[[9]], \mathbf{Kden}[[16]] - \mathbf{Kden}[[10]])$$

$$\left\{ \left\{ -\frac{3E}{8} - \frac{2H}{3}, 0, \frac{3E}{8} + \frac{2H}{3}, 0, \frac{E}{8} + \frac{H}{3}, 0, -\frac{E}{8} - \frac{H}{3}, 0, -\frac{3E}{8}, 0, \frac{3E}{8}, 0, \frac{E}{8}, 0, -\frac{E}{8}, 0 \right\}, \right. \\ \left\{ -\frac{E}{4} - \frac{H}{2}, 0, \frac{E}{4} + \frac{H}{2}, 0, \frac{E}{4} + \frac{H}{2}, 0, -\frac{E}{4} - \frac{H}{2}, 0, -\frac{E}{4}, 0, \frac{E}{4}, 0, \frac{E}{4}, 0, -\frac{E}{4}, 0 \right\}, \\ \left\{ 0, -\frac{2E}{17}, 0, -\frac{6E}{17}, 0, \frac{6E}{17}, 0, \frac{2E}{17}, 0, \frac{6E}{17}, 0, 0, 0, 0, 0, -\frac{6E}{17} \right\}, \\ \left\{ -\frac{E}{8} - \frac{H}{6}, 0, \frac{E}{8} + \frac{H}{6}, 0, -\frac{E}{8} - \frac{H}{6}, 0, \frac{E}{8} + \frac{H}{6}, 0, -\frac{E}{8}, 0, \frac{E}{8}, 0, -\frac{E}{8}, 0, \frac{E}{8}, 0 \right\}, \\ \left\{ 0, -\frac{12E}{17}, 0, -\frac{2E}{17}, 0, \frac{2E}{17}, 0, \frac{12E}{17}, 0, \frac{2E}{17}, 0, 0, 0, 0, 0, -\frac{2E}{17} \right\}, \left\{ -\frac{3E}{8}, 0, \frac{3E}{8}, 0, \frac{E}{8}, 0, -\frac{E}{8}, 0, \right. \\ \left. -\frac{3E}{8} - \frac{2H}{3}, 0, \frac{3E}{8} + \frac{2H}{3}, 0, \frac{E}{8} + \frac{H}{3}, 0, -\frac{E}{8} - \frac{H}{3}, 0 \right\}, \left\{ -\frac{E}{4}, 0, \frac{E}{4}, 0, \frac{E}{4}, 0, -\frac{E}{4}, 0, -\frac{E}{4} - \frac{H}{2}, \right. \\ \left. 0, \frac{E}{4} + \frac{H}{2}, 0, \frac{E}{4} + \frac{H}{2}, 0, -\frac{E}{4} - \frac{H}{2}, 0 \right\}, \left\{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\}, \\ \left\{ -\frac{E}{8}, 0, \frac{E}{8}, 0, -\frac{E}{8}, 0, \frac{E}{8}, 0, -\frac{E}{8} - \frac{H}{6}, 0, \frac{E}{8} + \frac{H}{6}, 0, -\frac{E}{8} - \frac{H}{6}, 0, \frac{E}{8} + \frac{H}{6}, 0 \right\}, \\ \left\{ 0, \frac{2E}{17}, 0, \frac{6E}{17}, 0, -\frac{6E}{17}, 0, -\frac{2E}{17}, 0, -\frac{6E}{17}, 0, 0, 0, 0, 0, \frac{6E}{17} \right\} \left. \right\}$$

Kstn2 = {
(Kstn1[[1, 3]] – Kstn1[[1, 1]], Kstn1[[1, 5]] – Kstn1[[1, 1]],
Kstn1[[1, 6]] – Kstn1[[1, 4]], Kstn1[[1, 7]] – Kstn1[[1, 1]],
Kstn1[[1, 8]] – Kstn1[[1, 2]], Kstn1[[1, 11]] – Kstn1[[1, 9]],
Kstn1[[1, 13]] – Kstn1[[1, 9]], Kstn1[[1, 14]] – Kstn1[[1, 12]],
Kstn1[[1, 15]] – Kstn1[[1, 9]], Kstn1[[1, 16]] – Kstn1[[1, 10]]),

(Kstn1[[2, 3]] – Kstn1[[2, 1]], Kstn1[[2, 5]] – Kstn1[[2, 1]],
Kstn1[[2, 6]] – Kstn1[[2, 4]], Kstn1[[2, 7]] – Kstn1[[2, 1]],
Kstn1[[2, 8]] – Kstn1[[2, 2]], Kstn1[[2, 11]] – Kstn1[[2, 9]],
Kstn1[[2, 13]] – Kstn1[[2, 9]], Kstn1[[2, 14]] – Kstn1[[2, 12]],
Kstn1[[2, 15]] – Kstn1[[2, 9]], Kstn1[[2, 16]] – Kstn1[[2, 10]]),

(Kstn1[[3, 3]] – Kstn1[[3, 1]], Kstn1[[3, 5]] – Kstn1[[3, 1]],
Kstn1[[3, 6]] – Kstn1[[3, 4]], Kstn1[[3, 7]] – Kstn1[[3, 1]],
Kstn1[[3, 8]] – Kstn1[[3, 2]], Kstn1[[3, 11]] – Kstn1[[3, 9]],
Kstn1[[3, 13]] – Kstn1[[3, 9]], Kstn1[[3, 14]] – Kstn1[[3, 12]],
Kstn1[[3, 15]] – Kstn1[[3, 9]], Kstn1[[3, 16]] – Kstn1[[3, 10]]),

(Kstn1[[4, 3]] – Kstn1[[4, 1]], Kstn1[[4, 5]] – Kstn1[[4, 1]],
Kstn1[[4, 6]] – Kstn1[[4, 4]], Kstn1[[4, 7]] – Kstn1[[4, 1]],
Kstn1[[4, 8]] – Kstn1[[4, 2]], Kstn1[[4, 11]] – Kstn1[[4, 9]],
Kstn1[[4, 13]] – Kstn1[[4, 9]], Kstn1[[4, 14]] – Kstn1[[4, 12]],
Kstn1[[4, 15]] – Kstn1[[4, 9]], Kstn1[[4, 16]] – Kstn1[[4, 10]]),

(Kstn1[[5, 3]] – Kstn1[[5, 1]], Kstn1[[5, 5]] – Kstn1[[5, 1]],
Kstn1[[5, 6]] – Kstn1[[5, 4]], Kstn1[[5, 7]] – Kstn1[[5, 1]],
Kstn1[[5, 8]] – Kstn1[[5, 2]], Kstn1[[5, 11]] – Kstn1[[5, 9]],
Kstn1[[5, 13]] – Kstn1[[5, 9]], Kstn1[[5, 14]] – Kstn1[[5, 12]],
Kstn1[[5, 15]] – Kstn1[[5, 9]], Kstn1[[5, 16]] – Kstn1[[5, 10]]),

(Kstn1[[6, 3]] – Kstn1[[6, 1]], Kstn1[[6, 5]] – Kstn1[[6, 1]],
Kstn1[[6, 6]] – Kstn1[[6, 4]], Kstn1[[6, 7]] – Kstn1[[6, 1]],
Kstn1[[6, 8]] – Kstn1[[6, 2]], Kstn1[[6, 11]] – Kstn1[[6, 9]],
Kstn1[[6, 13]] – Kstn1[[6, 9]], Kstn1[[6, 14]] – Kstn1[[6, 12]],
Kstn1[[6, 15]] – Kstn1[[6, 9]], Kstn1[[6, 16]] – Kstn1[[6, 10]]),

(Kstn1[[7, 3]] – Kstn1[[7, 1]], Kstn1[[7, 5]] – Kstn1[[7, 1]],
Kstn1[[7, 6]] – Kstn1[[7, 4]], Kstn1[[7, 7]] – Kstn1[[7, 1]],
Kstn1[[7, 8]] – Kstn1[[7, 2]], Kstn1[[7, 11]] – Kstn1[[7, 9]],
Kstn1[[7, 13]] – Kstn1[[7, 9]], Kstn1[[7, 14]] – Kstn1[[7, 12]],
Kstn1[[7, 15]] – Kstn1[[7, 9]], Kstn1[[7, 16]] – Kstn1[[7, 10]]),

(Kstn1[[8, 3]] – Kstn1[[8, 1]], Kstn1[[8, 5]] – Kstn1[[8, 1]],
Kstn1[[8, 6]] – Kstn1[[8, 4]], Kstn1[[8, 7]] – Kstn1[[8, 1]],
Kstn1[[8, 8]] – Kstn1[[8, 2]], Kstn1[[8, 11]] – Kstn1[[8, 9]],
Kstn1[[8, 13]] – Kstn1[[8, 9]], Kstn1[[8, 14]] – Kstn1[[8, 12]],
Kstn1[[8, 15]] – Kstn1[[8, 9]], Kstn1[[8, 16]] – Kstn1[[8, 10]]),

(Kstn1[[9, 3]] – Kstn1[[9, 1]], Kstn1[[9, 5]] – Kstn1[[9, 1]],
Kstn1[[9, 6]] – Kstn1[[9, 4]], Kstn1[[9, 7]] – Kstn1[[9, 1]],
Kstn1[[9, 8]] – Kstn1[[9, 2]], Kstn1[[9, 11]] – Kstn1[[9, 9]],
Kstn1[[9, 13]] – Kstn1[[9, 9]], Kstn1[[9, 14]] – Kstn1[[9, 12]],
Kstn1[[9, 15]] – Kstn1[[9, 9]], Kstn1[[9, 16]] – Kstn1[[9, 10]]),

(Kstn1[[10, 3]] – Kstn1[[10, 1]], Kstn1[[10, 5]] – Kstn1[[10, 1]],
Kstn1[[10, 6]] – Kstn1[[10, 4]], Kstn1[[10, 7]] – Kstn1[[10, 1]],
Kstn1[[10, 8]] – Kstn1[[10, 2]], Kstn1[[10, 11]] – Kstn1[[10, 9]],
Kstn1[[10, 13]] – Kstn1[[10, 9]], Kstn1[[10, 14]] – Kstn1[[10, 12]],
Kstn1[[10, 15]] – Kstn1[[10, 9]], Kstn1[[10, 16]] – Kstn1[[10, 10]])}

$$\left\{ \left\{ \frac{3E}{4} + \frac{4H}{3}, \frac{E}{2} + H, 0, \frac{E}{4} + \frac{H}{3}, 0, \frac{3E}{4}, \frac{E}{2}, 0, \frac{E}{4}, 0 \right\}, \left\{ \frac{E}{2} + H, \frac{E}{2} + H, 0, 0, 0, 0, \frac{E}{2}, \frac{E}{2}, 0, 0, 0 \right\}, \right. \\ \left. \left\{ 0, 0, \frac{12E}{17}, 0, \frac{4E}{17}, 0, 0, 0, 0, -\frac{12E}{17} \right\}, \left\{ \frac{E}{4} + \frac{H}{3}, 0, 0, \frac{E}{4} + \frac{H}{3}, 0, \frac{E}{4}, 0, 0, \frac{E}{4}, 0 \right\}, \right. \\ \left. \left\{ 0, 0, \frac{4E}{17}, 0, \frac{24E}{17}, 0, 0, 0, 0, -\frac{4E}{17} \right\}, \left\{ \frac{3E}{4}, \frac{E}{2}, 0, \frac{E}{4}, 0, \frac{3E}{4} + \frac{4H}{3}, \frac{E}{2} + H, 0, \frac{E}{4} + \frac{H}{3}, 0 \right\}, \right. \\ \left. \left\{ \frac{E}{2}, \frac{E}{2}, 0, 0, 0, \frac{E}{2} + H, \frac{E}{2} + H, 0, 0, 0 \right\}, \{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \}, \right. \\ \left. \left\{ \frac{E}{4}, 0, 0, \frac{E}{4}, 0, \frac{E}{4} + \frac{H}{3}, 0, 0, \frac{E}{4} + \frac{H}{3}, 0 \right\}, \left\{ 0, 0, -\frac{12E}{17}, 0, -\frac{4E}{17}, 0, 0, 0, 0, \frac{12E}{17} \right\} \right\}$$

(*This is to compute the eigenvalues of Kstn2*)

AbortProtect[Simplify[Eigenvalues[Kstn2]]]

$$\left\{ 0, 0, 0, 0, -\frac{4}{17}(-6 + \sqrt{2})E, \frac{4}{17}(6 + \sqrt{2})E, -\frac{1}{3}(-4 + \sqrt{7})H, \frac{1}{3}(4 + \sqrt{7})H, \right. \\ \left. \frac{1}{6}(9E + 8H - \sqrt{27E^2 + 54EH + 28H^2}), \frac{1}{6}(9E + 8H + \sqrt{27E^2 + 54EH + 28H^2}) \right\}$$