Appendix J

Variance of $w_{m,L}$

In this appendix, we derive the variance $\sigma_{m,L}^2$ of $w_{m,L}$, for $m \in \{1,2,3,\ldots,2^M\}$. It is defined as

$$\sigma_{m,L}^2 = E\{w_{m,L}w_{m,L}\} - \left(E\{w_{m,L}\}\right)^2 \tag{J.1}$$

Substituting $E\{w_{m,L}\} = \sqrt{2P}LT_cR_m(\tau)$ and (4.15) into (J.1) yields

$$\begin{aligned}\sigma_{m,L}^2 &= E\left\{\left(\sqrt{2P}LT_cR_m(\tau)\right)\left(\sqrt{2P}LT_cR_m(\tau)\right)\right\} + 2E\left\{\sqrt{2P}LT_cR_m(\tau)\eta_{m,L}\right\} \\ &\quad + E\{\eta_{m,L}\eta_{m,L}\} - \left(\sqrt{2P}LT_cR_m(\tau)\right)\left(\sqrt{2P}LT_cR_m(\tau)\right) \\ &= 2E\left\{\sqrt{2P}LT_cR_m(\tau)\eta_{m,L}\right\} + E\{\eta_{m,L}\eta_{m,L}\}.\end{aligned} \tag{J.2}$$

Since $E\{\eta_{m,L}\}$=0, $\sigma_{m,L}^2$ is equal to $E\{\eta_{m,L}\eta_{m,L}\}$, which can be computed as

$$\begin{aligned}E\{\eta_{m,L}\eta_{m,L}\} &= E\left\{\left[\sum_{i=0}^{L-1}\int_{iNT_c}^{(i+1)NT_c} \mathrm{Re}\{\eta(t)\}a_m(t)\,dt\right]\left[\sum_{j=0}^{L-1}\int_{jNT_c}^{(j+1)NT_c} \mathrm{Re}\{\eta(\lambda)\}a_m(\lambda)\,d\lambda\right]\right\} \\ &= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\int_{iNT_c}^{(i+1)NT_c}\int_{jNT_c}^{(j+1)NT_c} E\left\{\mathrm{Re}\{\eta(t)\}\mathrm{Re}\{\eta(\lambda)\}\right\}a_m(t)a_m(\lambda)\,d\lambda\,dt.\end{aligned} \tag{J.3}$$

The autocorrelation function of the real part of noise is

$$E\left\{\mathrm{Re}\{\eta(t)\}\mathrm{Re}\{\eta(\lambda)\}\right\} = N_0\delta(t-\lambda) \tag{J.4}$$

where $\delta(t)$ is a delta function. Substituting (J.4) into (J.3) and recalling that $a_m(t) = \sum_{i=0}^{N-1} a_{m,i}P_{T_c}(t - iT_c)$, we obtain

$$\begin{aligned}E\{\eta_{m,L}\eta_{m,L}\} &= N_0\sum_{i=0}^{L-1}\int_{iNT_c}^{(i+1)NT_c} a_m(t)a_m(t)\,dt \\ &= N_0LT_c\sum_{i=0}^{N-1} a_{m,i}a_{m,i}.\end{aligned} \tag{J.5}$$

Therefore, variance of $\eta_{m,L}$ can be expressed as

$$\sigma_{m,L}^2 = N_0LT_c\sum_{i=0}^{N-1} a_{m,i}a_{m,i}. \tag{J.6}$$

From Figure 4.3, the autocorrelation of $a_{m,i}$, $\mathrm{AC}(m) = \sum_{i=0}^{N-1} a_{m,i}a_{m,i}$, is a summation of two terms given below. Recall that $a_{m,i}$ comprises $(N-1)/2$ PN signals of different code phases with positively weighted coefficients and $(N+1)/2$ different PN signals of different code phases with negatively weighted coefficients.

(i) Correlation between $a_{m,i}$ and $(N-1)/2$ PN signals of different code phases with positively weighted coefficients. Let CC_+ denote this correlation. It can be computed as

$$\mathrm{CC}_+ = (N)\left(\frac{N-1}{2}\right)+(-1)\left[\left(\frac{N-1}{2}\right)\left(\frac{N-1}{2}\right)-\left(\frac{N-1}{2}\right)\right]+(-1)(-1)\left(\frac{N+1}{2}\right)\left(\frac{N-1}{2}\right) \quad \text{(J.7)}$$

$$= \frac{2N^2+2N-4}{4}. \quad \text{(J.8)}$$

The first and second terms of (J.7) are obtained from the correlations between $(N-1)/2$ positively weighted code phases of the PN signal due to $a_{m,i}$ and $(N-1)/2$ positively weighted code phases of the PN signal. Among them, there are $(N-1)/2$ terms with in-phase autocorrelation, contributing to the first term. The remainings, $\left[\left(\frac{N-1}{2}\right)\left(\frac{N-1}{2}\right)-\left(\frac{N-1}{2}\right)\right]$ terms produce out-of-phase autocorrelation values, contributing to the second term. The third term is obtained from the correlations between $(N+1)/2$ negatively weighted code phases of the PN signal due to $a_{m,i}$ and $(N-1)/2$ positively weighted code phases of the PN signal. These $\left(\frac{N+1}{2}\right)\left(\frac{N-1}{2}\right)$ terms give out-of-phase autocorrelation values. The negative sign (-1) in the third term is due to the product of negative and positive coefficients.

(ii) Correlation between $a_{m,i}$ and $(N+1)/2$ PN signals of different code phases with negatively weighted coefficients. Let CC_- denote the result. It can be computed as

$$\mathrm{CC}_- = (N)\left(\frac{N+1}{2}\right)+(-1)\left[\left(\frac{N+1}{2}\right)\left(\frac{N+1}{2}\right)-\left(\frac{N+1}{2}\right)\right]+(-1)(-1)\left(\frac{N-1}{2}\right)\left(\frac{N+1}{2}\right) \quad \text{(J.9)}$$

$$= \frac{2N^2+2N}{4}. \quad \text{(J.10)}$$

The terms in (J.9) are obtained similarly to (J.7). The first and second terms are from the correlations between $(N+1)/2$ negatively weighted code phases of the PN signal due to $a_{m,i}$ and $(N+1)/2$ negatively weighted code phases of the PN signal. The third term is from the correlations between $(N-1)/2$ positively weighted code phases of the PN signal due to $a_{m,i}$ and $(N+1)/2$ negatively weighted code phases of the PN signal.

Since $\mathrm{AC}(m) = \mathrm{CC}_+ + \mathrm{CC}_-$, substituting (J.8) and (J.10) into (J.6) yields

$$\sigma_{m,L}^2 = N_0 L T_c\left[\left(\frac{2N^2+2N-4}{4}\right)+\left(\frac{2N^2+2N}{4}\right)\right]$$
$$= N_0 L T_c\left(N^2+N-1\right). \quad \text{(J.11)}$$