Appendix L

Probability of Correct Estimate

The probability of correct estimate can be computed as follows. At specific τ, L, N and m^{th} (the branch number), the probability of correct estimate can be computed from

$$P_c(L|m,\tau) = \int_0^{\infty} f_m(w_{m,L}|\tau)\,dw_{m,L}, \tag{L.1}$$

for τ such that $R(\tau) > 0$, where $f_m(w_{m,L}|\tau)$ is defined in (4.17), and

$$P_c(L|m,\tau) = \int_{-\infty}^{0} f_m(w_{m,L}|\tau)\,dw_{m,L}. \tag{L.2}$$

for τ such that $R(\tau) < 0$.

Both can be expressed as

$$P_c(L|m,\tau) = \Phi\left(|R_m(\tau)|\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right), \tag{L.3}$$

where $|\cdot|$ is the absolute operator, SNR is the per-chip signal-to-noise ratio, which equals $2PT_c/N_0$, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ is the cumulative density function of the standard Gaussian random variable.

When there are M branches in the estimator, the statistics are independent between branches. Therefore, the probability of correct estimate given τ, $P_c(M,L|\tau)$ is

$$P_c(M,L|\tau) = \prod_{j=1}^{M}\Phi\left(|R_j(\tau)|\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right). \tag{L.4}$$

Consider when τ is located in the flat region which occupies $N-2^M$ chips. There are only two values of $\{R_m(\tau)\}$: $N+2$ and $-N$. $M-k$ branches possess $\{R_m(\tau)\} = -N$ and the remaining k branches possess $\{R_j(\tau)\} = N+2$. Hence, (L.4) can be expressed as

$$P_{c,\text{flat}}(M,L|\tau) = \left[\Phi\left(N\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^{M-k}\left[\Phi\left((N+2)\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^{k}. \tag{L.5}$$

Since there are M branches, there are 2^M sections to be chosen by the estimator. Consequently, there are $\binom{M}{k} = \frac{M!}{(M-k)!k!}$ combinations containing $M-k$ of negative binary value and k of the positive binary value for a binary sequence of M bits. Therefore, since the order of bits in the sequence is not important, there are $\binom{M}{k}$ sections offering (L.5). Of the 2^M sections in the uncertainty region, there is one section that consists of

$\frac{N-(2^{M+1}-1)}{2^M}$ cells and 2^M-1 sections that contain $\frac{N-(2^M-1)}{2^M}$ cells. The only section with a different number of cells is specified by a binary sequence corresponding to $\{R_m(\tau)\}=N+2$ for m=1, 2,..., M. Recall that there are $N-2^M$ cells located in the flat region. The average probability of correct estimate is obtained by averaging (L.5) over τ in the flat region, which is equivalent to the summation of binomial probability law, i.e.,

$$P_{c,\text{flat}}(M,L)=\int_{\tau\in\text{flat region}} P_{c,\text{flat}}(M,L|\tau)f(\tau)d\tau$$

$$=\left(\frac{N-(2^M-1)}{2^M N}\right)\sum_{k=0}^{M-1}\left\{\binom{M}{k}\left[\Phi\left(N\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^{M-k}\left[\Phi\left((N+2)\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^k\right\}$$

$$+\left(\frac{N-(2^{M+1}-1)}{2^M N}\right)\left[\Phi\left((N+2)\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^M, \qquad \text{(L.6)}$$

where $f(\tau)=1/NT_c$. The second term corresponds to the only section that has a different number of cells.

Consider the case when τ is located in the transition region which occupies 2^M chips. Within each transition cell, one branch has a varying $R_m(\tau)$, the other $M-1$ branches has a constant $\{R_j(\tau)\}$, with a value of $N+2$ or $-N$. For τ in the cell with one branch having a varying $R_m(\tau)$, the other $M-1$ branch can have 2^{M-1} combinations of $(M-1)$-bit binary sequence. Among the 2^{M-1} sequences, there are $\binom{M-1}{k}$ sequences possessing $M-1-k$ bits of one binary value and k bits of the other value. Therefore, the probability of correct estimate when τ is located at transition region is

$$P_{c,\text{tr}}(M,L|\tau)=\Phi\left(|R(\tau)|\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)$$

$$\cdot\left[\Phi\left(N\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^{M-k-1}\left[\Phi\left((N+2)\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^k. \qquad \text{(L.7)}$$

Referring to Figure 4.3, there are two kinds of varying $R_m(\tau)$: $R_m(\tau)$ changing from $N+2$ to $-N$ and $R_m(\tau)$ changing from $-N$ to $N+2$. Therefore, from (L.7), the average probability of correct estimate for the first kind is

$$P_{c,\text{tr1}}(M,L|\tau)=\int_{-0.5+\frac{1}{2(N+1)}}^{0.5+\frac{1}{2(N+1)}}\Phi\left(|2(N+1)v|\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)dv$$

$$\cdot\left[\Phi\left(N\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^{M-k-1}\left[\Phi\left((N+2)\sqrt{\frac{L\cdot\text{SNR}}{N^2+N-1}}\right)\right]^k. \qquad \text{(L.8)}$$

The fraction $\frac{1}{2(N+1)}$ is due to asymmetric transition from $N+2$ to $-N$. The average probability of correct estimate for the second kind is similar to (L.8), so that

$$P_{c,\mathrm{tr2}}\left(M,L|\tau\right)=P_{c,\mathrm{tr1}}\left(M,L|\tau\right). \tag{L.9}$$

For later use, we rewrite (L.8) as

$$P_{c,\mathrm{tr1}}\left(M,L|\tau\right)=$$

$$\left[\int_0^{0.5+\frac{1}{2(N+1)}}\Phi\left(2(N+1)v\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)dv+\int_0^{0.5-\frac{1}{2(N+1)}}\Phi\left(2(N+1)v\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)dv\right]$$

$$\cdot\left[\Phi\left(N\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)\right]^{M-k-1}\left[\Phi\left((N+2)\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)\right]^{k}. \tag{L.10}$$

The first integration corresponds to the average over positive $R_m(\tau)$, $R_m(\tau)\in[0,\,N+2)$, while the second integration corresponds to the average over negative $R_m(\tau)$, $R_m(\tau)\in(-N,\,0]$.

The average probability of correct estimate for τ in the transition region is obtained by averaging (L.9) and (L.10) over τ in the transition region, yielding

$$P_{c,\mathrm{tr}}(M,L)=\int_{\tau\in\mathrm{tr1\ region}}P_{c,\mathrm{tr1}}\left(M,L|\tau\right)f(\tau)d\tau+\int_{\tau\in\mathrm{tr2\ region}}P_{c,\mathrm{tr2}}\left(M,L|\tau\right)f(\tau)d\tau$$

$$=\left[\int_0^{0.5+\frac{1}{2(N+1)}}\Phi\left(2(N+1)v\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)dv+\int_0^{0.5-\frac{1}{2(N+1)}}\Phi\left(2(N+1)v\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)dv\right]$$

$$\cdot\left(\frac{2}{N}\right)\sum_{k=0}^{M-1}\binom{M-1}{k}\left[\Phi\left(N\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)\right]^{M-k-1}\left[\Phi\left((N+2)\sqrt{\frac{L\cdot \mathrm{SNR}}{N^2+N-1}}\right)\right]^{k}. \tag{L.11}$$

Eventually, the probability of correct estimate is obtained from combining (L.6) and (L.11),

$$P_c(M,L)=P_{c,\mathrm{flat}}(M,L)+P_{c,\mathrm{tr}}(M,L). \tag{L.12}$$