CHAPTER 3

THEORETICAL BACKGROUND

3.1 Smeared Crack Model

According to the fundamental of the smeared crack approach, an additional strain, called the crack strain, represents the existence of cracks in the cracked solid. Cracks do not appear as a separated boundary but they are represented by modifying the constitutive law (stress-strain relationship) of the solid intersected by the crack. By using the finite element method based on the smeared crack model, the cracked solid is modeled as a degraded or cracked element without need for special interface on the element. Consequently, the remeshing of the finite element mesh is unnecessary. However, there are some disadvantages on the assumption of continuity those conflicts with the reality of discontinuity of displacement field over the cracks.

In the smeared crack model, the total strain increment $\Delta\varepsilon$ is decomposed into the strain increment of the intact elastic solid $\Delta\varepsilon^{o}$ and the strain increment of the cracked solid $\Delta\varepsilon^{cr}$ (de Borst and Nauta 1985), i.e.,

$$\Delta\varepsilon = \Delta\varepsilon^{o} + \Delta\varepsilon^{cr}. \tag{3.1}$$

In the local coordinates in two-dimensional problems, the local crack strain increment $\Delta\hat{\varepsilon}^{cr}$ is written as

$$\Delta\hat{\varepsilon}^{cr} = \left(\Delta\hat{\varepsilon}_{nn}^{cr} \quad \Delta\hat{\gamma}_{nt}^{cr}\right)^{T} \tag{3.2}$$

in which $\Delta\hat{\varepsilon}_{nn}^{cr}$ and $\Delta\hat{\gamma}_{nt}^{cr}$ are the mode I normal and mode II shear crack strain increments, respectively. The relationship between the global crack strain increment vector $\Delta\varepsilon^{cr}$ and local crack strain increment vector $\Delta\hat{\varepsilon}^{cr}$ can be written as

$$\Delta\varepsilon^{cr} = \mathbf{T}\Delta\hat{\varepsilon}^{cr} \tag{3.3}$$

which $\mathbf{T}$ is the transformation matrix between the global and local coordinate systems is defined, for two-dimensional problems, as

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ \sin^2\theta & \sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{3.4}$$

where θ is the angle between the vector normal to the crack surfaces and the global x-axis (see Fig. 3.1).

In the local coordinate system, we also have the local traction increment $\Delta\hat{\mathbf{t}}^{cr}$ across the crack defined as

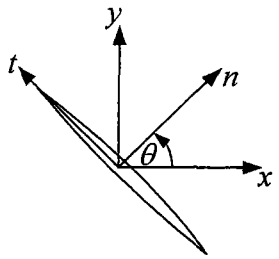


Fig. 3.1 Crack angle θ

$$\Delta\hat{\mathbf{t}}^{cr} = \left[\Delta\hat{t}_n^{cr} \quad \Delta\hat{s}_t^{cr}\right]^T \tag{3.5}$$

when $\Delta\hat{t}_n^{cr}$ and $\Delta\hat{s}_t^{cr}$ are the normal and shear crack traction increments, respectively.

The relationship between the local traction increment and local strain increment is written as

$$\Delta\hat{\mathbf{t}}^{cr} = \hat{\mathbf{D}}^{cr}\Delta\hat{\boldsymbol{\varepsilon}}^{cr} \tag{3.6}$$

where $\hat{\mathbf{D}}^{cr}$ is the crack constitutive matrix (Rots 1989, De Borst and Nauta 1985) written for two-dimensional problems as

$$\hat{\mathbf{D}}^{cr} = \begin{bmatrix} D^{I} & 0 \\ 0 & D^{II} \end{bmatrix}. \tag{3.7}$$

Here, D^{I} and D^{II} are the mode I and mode II crack modulus, respectively. The local traction increment $\Delta\hat{\mathbf{t}}^{cr}$ is related to the global stress increment $\Delta\sigma$ as

$$\Delta\hat{\mathbf{t}}^{cr} = \mathbf{T}^{T}\Delta\sigma\,. \tag{3.8}$$

The constitutive relationship between the global stress increment $\Delta\sigma$ and the global strain increment of the uncracked solid $\Delta\boldsymbol{\varepsilon}^{o}$ is given as

$$\Delta\sigma = \mathbf{D}^{o}\Delta\varepsilon^{o}\,. \tag{3.9}$$

The construction of the smeared crack stiffness begins with the constitutive equation of the uncracked solid in Eq. (3.9). Using Eqs. (3.1) and (3.3) in Eq. (3.9), we get

$$\Delta\boldsymbol{\sigma} = \mathbf{D}^{o}\Delta\boldsymbol{\varepsilon} - \mathbf{D}^{o}\mathbf{T}\Delta\hat{\boldsymbol{\varepsilon}}^{cr}\,. \tag{3.10}$$

Substituting Eq. (3.9) into Eq. (3.8) and using Eq. (3.1), we write the local traction increment as

$$\Delta\hat{\mathbf{t}}^{cr} = \mathbf{T}^{T}\mathbf{D}^{o}\left(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{cr}\right). \tag{3.11}$$

Substituting Eq. (3.11) into Eq. (3.6) and using Eq. (3.3), we obtain the relationship between the local crack strain increment $\Delta\hat{\boldsymbol{\varepsilon}}^{cr}$ and total strain increment $\Delta\boldsymbol{\varepsilon}$ as

$$\Delta\hat{\boldsymbol{\varepsilon}}^{cr} = \left(\hat{\mathbf{D}}^{cr} + \mathbf{T}^T\mathbf{D}^o\mathbf{T}\right)^{-1}\mathbf{T}^T\mathbf{D}^o\Delta\boldsymbol{\varepsilon}. \tag{3.12}$$

Substituting Eq. (3.12) into Eq. (3.10), we obtain the overall stress-strain relationship (De Borst and Nauta 1985),i.e.,

$$\Delta\boldsymbol{\sigma} = \left[\mathbf{D}^o - \mathbf{D}^o\mathbf{T}\left(\mathbf{D}^{cr} + \mathbf{T}^T\mathbf{D}^o\mathbf{T}\right)^{-1}\mathbf{T}^T\mathbf{D}^o\right]\Delta\boldsymbol{\varepsilon} = \overline{\mathbf{D}}\Delta\boldsymbol{\varepsilon} \tag{3.13}$$

which is the constitutive law for the cracked material in the smeared crack model. Here, $\overline{\mathbf{D}}$ is the constitutive matrix of the crack material.

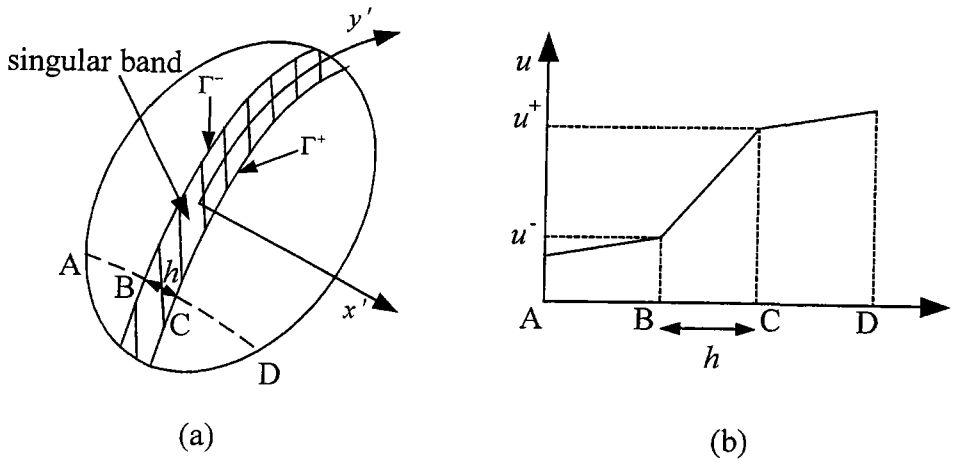
3.2 Characteristic Length in The Smeared Crack Model

One of the important parameters in the smeared crack model is the characteristic length. The simple definition of the characteristic length is the width of the damaged strip where the crack constitutive law is satisfied. In the crack band model proposed by Bazant and Oh (1983), the characteristic length or the crack band width is considered as one of the material properties and it is considered to be dependent on the maximum aggregate size. This concept is not suitable for the finite element computation, especially for curved crack paths. The reason is that the boundary of the crack band may be different from the boundary of the existing elements. This will lead to different material properties within one element. Another idea is to use the boundary of elements as boundary of crack band instead of using fixed crack band width. This implies that the crack band width is not considered as a material property. In this case, the size of the smeared crack band will depend on mesh size. Without careful consideration, the results of the computation will be mesh dependent. The objectivity of the results can be achieved by modifying the constitutive law and making it depends on mesh size.

Oliver (1989) modeled a two-dimensional crack as a limiting case of two singular lines, which tend to coincide with each other (Fig. 3.2a). Across two lines, the displacements are continuous but the displacement gradients are discontinuous (Fig. 3.2b). Following his idea, the characteristic length l^* is defined as the ratio between the energy dissipated per unit surface area (fracture energy) G_f and the energy dissipated per unit volume (specific energy) g_f at a point, i.e.,

$$l^* = \frac{G_f}{g_f} \tag{3.14}$$

where the energy dissipation represented by G_f and g_f is from time $t = 0$ to $t = \infty$.



(a) (b)

Fig. 3.2 A crack as a singular band between two singular lines (Oliver 1987)

Consider a crack band or a singular band, bounded by the two singular lines, in a solid with an infinitesimal width h (see Fig. 3.2). Along a coordinate line x', the displacement vector $\mathbf{u}$ may be expanded from its value in the Γ^- line as

$$\mathbf{u}(x',y')=\mathbf{u}^-(y')+\left[\frac{\partial \mathbf{u}}{\partial x'}\right]^- \Delta x'+O(h^2) \tag{3.15}$$

and for a point in the Γ^+ line

$$\mathbf{u}^+(y')=\mathbf{u}^-(y')+\left[\frac{\partial \mathbf{u}}{\partial x'}\right]^- h(y')+O(h^2) \tag{3.16}$$

where the superscript + and – denote values in the Γ^+ and Γ^- lines, respectively.

From Eqs. (3.15) and (3.16), we write

$$\mathbf{u}(x',y')=\mathbf{u}^-(y')+\frac{\Delta x'}{h}\left[\mathbf{u}^+(y')-\mathbf{u}^-(y')\right]+O(h^2)\approx \mathbf{u}^-(y')+\phi(x',y')\left[\mathbf{u}^+(y')-\mathbf{u}^-(y')\right] \tag{3.17}$$

where ϕ is a function that approximate $\dfrac{\Delta x'}{h}$ when $h\to 0$. It can be seen that

$$\phi=0 \text{ on } \Gamma^- \text{ and } \phi=1 \text{ on } \Gamma^+. \tag{3.18}$$

By considering the energy dissipation within the band and the equilibrium across the band, it can be shown that (Oliver 1989)

$$g_f=G_f\frac{\partial \phi}{\partial x'}=\frac{G_f}{l^*} \tag{3.19}$$

where l^* is the characteristic length given by

$$l^*(x',y')=\left(\frac{\partial\phi}{\partial x'}\right)^{-1}. \tag{3.20}$$

Consider a mesh of C^0 elements in Fig. 3.3. The function ϕ has to be continuous, derivable function satisfying Eq. (3.18). Therefore, the function defined in the natural coordinate system can be used, i.e.,

$$\phi(\xi,\eta)=\sum_{i=1}^{n_c} N_i^*(\xi,\eta)\phi_i \tag{3.21}$$

which n_c is the number of corner nodes of the elements, and N_i^* are the standard shape functions of an element of n_c nodes. Here, ϕ_i represents the value of the function ϕ at corner node i. As discussed earlier, a two-dimensional crack is modeled as a limiting case of two singular lines, which tend to coincide with each other. If the location of the crack within the element is know, ϕ_i will take the value +1 for each node on one side of the crack and 0 on the other side [Eq. (3.18)].

From Eqs. (3.20) and (3.21), the characteristic length is obtained as

$$l^*(\xi,\eta)=\left(\sum_{i=1}^{n_c}\left[\frac{\partial N_i^*(\xi,\eta)}{\partial x}\cos\theta+\frac{\partial N_i^*(\xi,\eta)}{\partial y}\sin\theta\right]\phi_i\right)^{-1} \tag{3.22}$$

where θ is the angle between the normal to the crack and the global x-axis, and ξ and η are the natural coordinates (see Fig. 3.3).

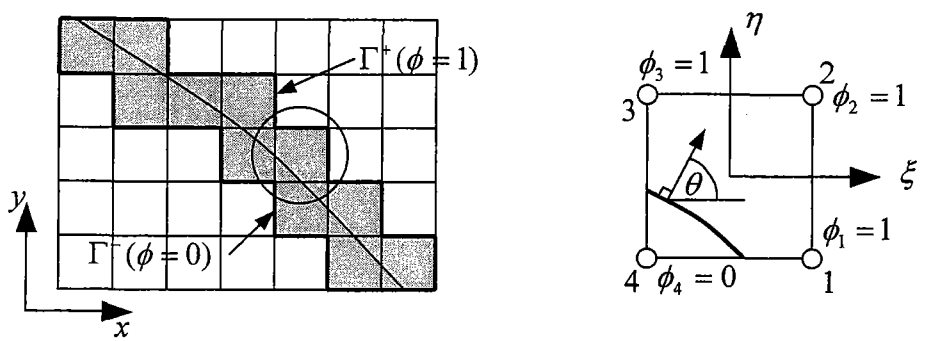


Fig. 3.3 A singular band in a finite element mesh (Oliver 1987)

3.3 Localization and Bifurcation

Following the concept proposed by Nguyen (1987) and Brocca (1997), we consider a system of a deformable body with cracks where the energy is dissipated. The total energy of the body is defined as

$$\Pi(u_i,\alpha_j)=\Pi^M(u_k,\alpha_l)+\Pi^D(\alpha_m) \tag{3.23}$$

where $\Pi^M(u_k,\alpha_l)$ is the mechanical potential energy and the $\Pi^D(\alpha_m)$ is the dissipated energy. Here, u_i's ($i=1,...,N$) represent reversible variables and α_j's ($j=1,...,K$) represent irreversible variables. Here, N and K are the number of reversible variables and the number of irreversible variables, respectively. The irreversible variables characterize the inelastic behavior of the materials, i.e., a crack opening displacement in the discrete crack approach or a crack strain in the smeared crack approach. Additionally, the mechanical potential energy includes the strain energy and the external potential energy while the dissipated energy is the energy transformed from mechanical energy in the system to thermal energy through irreversible process.

The fundamental solution, obtained by applying the stationary condition to Eq. (3.23), is written as

$$\begin{aligned} &\frac{\partial\Pi}{\partial u_i}=0, \quad (i=1,\ldots,N) \\ &\frac{\partial\Pi}{\partial\alpha_j}=0, \quad (j=1,\ldots,K). \end{aligned} \tag{3.24}$$

By employing the equilibrated solution in Eq. (3.24), the reversible variables are expressed in terms of the irreversible variables, i.e., $u_i=u_i(\alpha_j)$. Consequently, the total energy of the body in Eq. (3.23) can be expressed as a functional of only the irreversible variables, i.e.,

$$\Pi^*(\alpha_l)=\Pi^{*M}(\alpha_j)+\Pi^D(\alpha_m) \tag{3.25}$$

where $\Pi^*(\alpha_l)=\Pi(u_i(\alpha_k),\alpha_j)$ and $\Pi^{*M}(\alpha_j)=\Pi^M(u_k(\alpha_m),\alpha_l)$.

The signs of the eigenvalues of the Hessian matrix $\left[\frac{\partial^2\Pi^*}{\partial\alpha_i\partial\alpha_j}\right]$ are can be used to investigate the stability of the fundamental solution obtained from Eq. (3.24). If all the eigenvalues are positive, the fundamental solution is stable. Otherwise, the fundamental solution is unstable and the localization occurs.

3.4 Genetic Algorithms

In order to obtain the actual equilibrium path, all of the possible crack patterns can be examined (exhaustive search) in order to search for the one with the minimum total potential energy. However, this exhaustive search method is very expensive and not practical. Therefore, a powerful and more practical search algorithm is necessary to reduce the computational time.

The genetic algorithm is a computerized search algorithm. The algorithm is based on the mechanic of natural genetics and natural selection, called the survival of the fittest (Holland 1975). It is becoming popular recently because it seems to be a potential search and optimization algorithm for the complex engineering optimization problems. In this study, the genetic algorithm is used as a search algorithm for finding an equilibrium path that has the minimum total potential energy. The algorithm is selected because it is suitable for optimization problems with discrete variables. The variables under consideration in the search for the path with the minimum energy are the statuses of cracks that can be either opening or unloading. These variables are clearly discrete.

The idea of genetic algorithms is that each search solution is assumed to be a life form which is composed of the genetic data indicating an ability of the life form, called a fitness value. According to the natural selection, the life form with a high ability has more chance to survive. After many generations, good genetic codes will be duplicated from parents to their children. Thus, the populations in later generations are expected to be better than their ancestors. Finally, the life form with the best ability can be expected.

The algorithm starts with specifying an objective function and its variables. As an example, consider the following optimization problem, i.e.,

$$\text{Maximize } f(\mathbf{x})$$

where $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_N) \in \mathbf{R}^N$ and $x_i^L \le x_i \le x_i^U$, $i = 1, \ldots, N$.

Although a maximization problem is considered here, genetic algorithms can also handle a minimization problem. The working of genetic algorithms is completed by performing the following tasks.

3.4.1 Coding

In order to solve the problem, variable x_i's are first coded in some string structures. Binary-coded strings having 1's and 0's are mostly used. The length of the string is usually determined by the accuracy of a solution. For example, if four bits are used to code each variable in a two-variable function optimization problem, the string (0000 0000) and (1111 1111) will represent the points $\left(x_1^{(L)}, x_2^{(L)}\right)$ and $\left(x_1^{(U)}, x_2^{(U)}\right)$, respectively. The substrings (0000) and (1111) are interpreted as the minimum and maximum decoded values. The decoding can be done by the traditional transformation function. In this above example, it is worthwhile to mention here that with four bits to code each variable, there are 2^4 or 16 distinct substrings possible

because each bit-position can take a value either 0 or 1. The accuracy that can be obtained with a four-bit coding is $1/16^{th}$ of the search space. It is not necessary to code all variables in equal substring length. The length of a substring representing a variable depends on the desired accuracy in that variable. Once the coding of the variables has been done, the corresponding point $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ in search space can be found. Thereafter, the function value at the point $\mathbf{x}$ can also be calculated by substituting $\mathbf{x}$ in the objective function $f(\mathbf{x})$.

3.4.2 Fitness function

As pointed out earlier, genetic algorithms mimic the survival-of-the-fittest principle of nature to make a search process. Therefore, genetic algorithms are naturally suitable for solving maximization problems. For maximization problems, the fitness function $F(\mathbf{x})$ can be considered as an objective function or $F(\mathbf{x}) = f(\mathbf{x})$. For minimization problems, their objective functions are usually transformed into fitness functions by some suitable transformation (Deb 1995). The following fitness function is often used, i.e.,

$$F(\mathbf{x}) = \frac{1}{(1 + f(\mathbf{x}))}. \tag{3.26}$$

Note that this transformation does not alter the location of the minimum, but converts a minimization problem to an equivalent maximization problem. The fitness function value of a string is known as the string's fitness.

The procedure of genetic algorithms begins with a population of randomized strings representing design or decision variables. Thereafter, each string is evaluated to find the fitness value. The population is manipulated by three main operators—reproduction, crossover and mutation operators—to create a new population of points. The new population is further evaluated and tested for termination. If the termination criterion is not met, the population is iteratively operated by the above three operators and evaluated. The procedure is continued until the termination criterion is met. One cycle of these operations and the subsequent evaluation procedure is known as a *generation* in genetic algorithm's terminology.

3.4.3 Genetic algorithm's operators

Reproduction is usually the first operator applied on a population. Reproduction operator selects good strings from a population and forms a mating pool. The essential idea is the high fitness strings are picked from the current population and their multiple copies are inserted into the mating pool in a probabilistic manner. The commonly used reproduction operator is the proportionate reproduction operator where a string is selected for the mating pool with a probability proportional to its fitness. Thus, the i^{th} string in the population is selected with a probability proportional to its stiffness F_i, i.e.,

$$p_i = \frac{F_i}{\sum_{j=1}^{n} F_j} \tag{3.27}$$

where n is the population size.

In the crossover operator, new strings are created by exchanging the information among strings in the mating pool. Many crossover operators exist in the genetic algorithms literature (Goldberg 1989, and Deb 1995). In most crossover operators, two strings are picked from the mating pool at random and some portions of the strings are exchanged between the strings. The two strings participating in the crossover operation are known as parent strings and the resulting strings are known as children strings. For example, a single-point crossover operator is performed by randomly choosing a crossing site along the string and by exchanging all bits on the right side of the crossing site. It is intuitive from this construction that good substrings from parent strings can be combined to form better children strings, if an appropriate crossing site is chosen.

Although a crossover operator is mainly responsible for the search of new strings, a mutation operator is also used for this purpose sparingly. In common mutation operator, each bit in population is picked up and generates a randomized value is generated. If such value is lower than mutation probability (usually small probability), the mutation operator will be performed by changing the bit value from 1 to 0 and vice versa.

These three operators are simple and straightforward. The reproduction operator selects good strings and the crossover operator combines good substrings together to hopefully create better substrings. The mutation operator alters a string locally to hopefully create a better string. Even though none of these claims are guaranteed and/or tested while creating a string, it is expected that if bad strings are created they will be eliminated by the reproduction operator in the next generation and if good strings are created, they will be increasingly emphasized.