CHAPTER 5

MIXED FINITE ELEMENT FORMULATION FOR THE SMEARED CRACK MODEL

In order to determine the stability of crack patterns, it is necessary to obtain the second variation of energy of the system with respect to irreversible crack variables. In other words, the stability of the equilibrium with respect to irreversible crack variables must be investigated. If the smeared crack model is used in the analysis, the irreversible crack variable is the crack strain. However, in the smeared crack model, the crack strain is not discretized and is actually a function of position. Therefore, it is difficult to perform the stability analysis with respect to this non-discretized variable. To circumvent this problem, a mixed formulation that discretizes not only the displacement field but also the crack strain field can be used (Nanakorn and Soparat 2000). The discretized crack strain variable will allow the stability of the system with respect to the irreversible crack strain to be done easily and efficiently.

5.1 The Mixed Finite Element Formulation

By following Nanakorn and Soparat (2000), the total potential energy increment of a cracked domain V is expressed as

$$\begin{aligned}\Delta\Pi &= \Delta\Pi^{M} + \Delta\Pi^{D} \\ &= \left[\frac{1}{2}\int_V \Delta\varepsilon^{o^T}\Delta\sigma\, dV - \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dV - \int_S \Delta\mathbf{u}^T\Delta\mathbf{t}dS\right] + \left[\frac{1}{2}\int_V \Delta\hat{\varepsilon}^{cr^T}\Delta\hat{\mathbf{t}}^{cr}dV\right].\end{aligned} \tag{5.1}$$

The total potential energy increment $\Delta\Pi$ shown above consists of two parts that are the mechanical potential energy increment $\Delta\Pi^M$ and the dissipated energy increment $\Delta\Pi^D$. Here, $\Delta\mathbf{u}$ denotes the displacement increment vector. In addition, $\Delta\mathbf{f}$ and $\Delta\mathbf{t}$ are the body force increment vector and the surface traction increment vector, respectively. By substituting Eqs. (3.6) and (3.9) into Eq. (5.1), the total potential energy increment is written as

$$\begin{aligned}\Delta\Pi &= \Delta\Pi^{M} + \Delta\Pi^{D} \\ &= \left[\frac{1}{2}\int_V \Delta\varepsilon^{o^T}\mathbf{D}^o\Delta\varepsilon^o\, dV - \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dV - \int_S \Delta\mathbf{u}^T\Delta\mathbf{t}dS\right] + \left[\frac{1}{2}\int_V \Delta\hat{\varepsilon}^{cr^T}\hat{\mathbf{D}}^{cr}\Delta\hat{\varepsilon}^{cr}dV\right].\end{aligned} \tag{5.2}$$

In the expression of the total potential energy increment in Eq. (5.2), the irreversible variable that has to be considered in the stability analysis is the local crack strain increment $\Delta\hat{\varepsilon}^{cr}$. The first variation of the energy increment with respect to this local crack strain increment results in the equilibrium path. The second variation will give the information on the stability condition of the obtained equilibrium path. Since the total energy increment is a functional of the crack strain increment function, the calculus of variations is required. To avoid this difficulty, a mixed formulation in the finite element method by discretizing both displacement increment field and the local

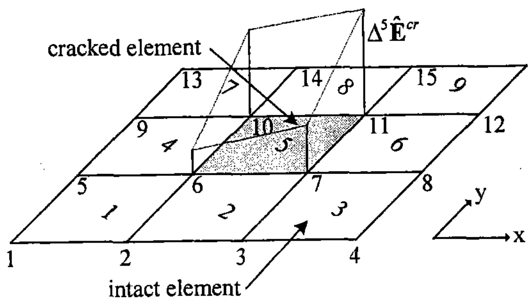


Fig. 5.1 A problem with one cracked element surrounded by intact elastic elements (Nanakorn and Soparat 2000)

crack strain increment field is employed. For the i^{th} element in the finite element analysis, we have

$$\Delta\mathbf{u} = \mathbf{N}\Delta\mathbf{U} \tag{5.3a}$$

$$\Delta^i\hat{\varepsilon}^{cr} = \mathbf{N}^{cr}\Delta^i\hat{\mathbf{E}}^{cr} \tag{5.3b}$$

where $\mathbf{N}$ and $\mathbf{N}^{cr}$ represent the shape function matrices for the displacement increment and the local crack strain increment, respectively. Moreover, $\Delta\mathbf{U}$ and $\Delta^i\hat{\mathbf{E}}^{cr}$ represent the nodal displacement increment and the nodal crack strain increment, respectively. Note that the local crack strain increments are not continuous across elements and the nodal local crack strain increments of the same node for different elements can be different. One example is a problem with one cracked element surrounded by uncracked elements (see Fig. 5.1). In the cracked element including its boundary, non-zero crack strain increments can be expected. However, in the surrounding uncracked elements, the crack strain increments are expected to be zero because there is no crack in those elements. On the contrary, the total displacement increments must be continuous across all elements.

Substituting Eq. (3.1) in Eq. (5.2), we write the total energy increment for the i^{th} element as

$$\begin{aligned}\Delta\Pi = &\frac{1}{2}\int_V (\Delta\varepsilon - \Delta\varepsilon^{cr})^T \mathbf{D}^o (\Delta\varepsilon - \Delta\varepsilon^{cr}) dV + \frac{1}{2}\int_V \Delta^i\hat{\varepsilon}^{cr^T}\hat{\mathbf{D}}^{cr}\Delta^i\hat{\varepsilon}^{cr} dV \\ &- \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dV - \int_S \Delta\mathbf{u}^T\Delta t dS\end{aligned} \tag{5.4}$$

which yields

$$\begin{aligned}\Delta\Pi = &\frac{1}{2}\int_V \Delta\varepsilon^T\mathbf{D}^o\Delta^i\varepsilon dV - \frac{1}{2}\int_V \Delta\varepsilon^T\mathbf{D}^o\Delta^i\varepsilon^{cr} dV - \frac{1}{2}\int_V \Delta^i\varepsilon^{cr^T}\mathbf{D}^o\Delta\varepsilon dV \\ &+ \frac{1}{2}\int_V \Delta^i\varepsilon^{cr^T}\mathbf{D}^o\Delta^i\varepsilon^{cr} dV + \frac{1}{2}\int_V \Delta^i\hat{\varepsilon}^{cr^T}\hat{\mathbf{D}}^{cr}\Delta^i\hat{\varepsilon}^{cr} dV \\ &- \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dV - \int_S \Delta\mathbf{u}^T\Delta\mathbf{f}dS.\end{aligned} \tag{5.5}$$

From Eqs. (3.3) and (5.3), the total strain increment and the global crack strain increment are expressed as

$$\Delta\varepsilon = \mathbf{B}\Delta\mathbf{U}, \tag{5.6a}$$
$$\Delta^i\varepsilon^{cr} = \mathbf{T}\mathbf{N}^{cr}\Delta^i\hat{\mathbf{E}}^{cr}. \tag{5.6b}$$

From Eqs. (3.1), (5.4) and (5.6), the total potential energy increment can be expressed as

$$\begin{aligned}\Delta\Pi = &\frac{1}{2}\int_V \Delta^i\mathbf{U}^T\mathbf{B}^T\mathbf{D}^o\mathbf{B}\Delta^i\mathbf{U}dV - \frac{1}{2}\int_V \Delta^i\mathbf{U}^T\mathbf{B}^T\mathbf{D}^o\mathbf{T}\mathbf{N}^{cr}\Delta^i\hat{\mathbf{E}}^{cr}dV \\ &-\frac{1}{2}\int_V \Delta^i\hat{\mathbf{E}}^{cr^T}\mathbf{N}^{cr^T}\mathbf{T}^T\mathbf{D}^o\mathbf{B}\Delta^i\mathbf{U}dV + \frac{1}{2}\int_V \Delta^i\hat{\mathbf{E}}^{cr^T}\mathbf{N}^{cr^T}\mathbf{T}^T\mathbf{D}^o\mathbf{T}\mathbf{N}^{cr}\Delta^i\hat{\mathbf{E}}^{cr}dV \\ &+\frac{1}{2}\int_V \Delta^i\hat{\mathbf{E}}^{cr^T}\mathbf{N}^{cr^T}\hat{\mathbf{D}}^{cr}\mathbf{N}^{cr}\Delta^i\hat{\mathbf{E}}^{cr}dV - \int_V \Delta^i\mathbf{U}^T\mathbf{N}^T\Delta\mathbf{f}dV - \int_S \Delta^i\mathbf{U}^T\mathbf{N}^T\Delta\mathbf{t}dS.\end{aligned} \tag{5.7}$$

Applying the stationary condition $\delta(\Delta\Pi)=0$ to Eq. (5.7) and assuming that $\mathbf{D}^o$ and $\hat{\mathbf{D}}^{cr}$ are symmetry, we get

$$\begin{aligned}\delta(\Delta\Pi) = &\ \delta\left(\Delta^i\mathbf{U}^T\right)\int_V \mathbf{B}^T\mathbf{D}^o\mathbf{B}dV\Delta^i\mathbf{U} - \delta\left(\Delta^i\mathbf{U}^T\right)\int_V \mathbf{B}^T\mathbf{D}^o\mathbf{T}\mathbf{N}^{cr}dV\Delta^i\hat{\mathbf{E}}^{cr} \\ &-\delta\left(\Delta^i\hat{\mathbf{E}}^{cr^T}\right)\int_V \mathbf{N}^{cr^T}\mathbf{T}^T\mathbf{D}^o\mathbf{B}dV\Delta^i\mathbf{U} + \delta\left(\Delta^i\hat{\mathbf{E}}^{cr^T}\right)\int_V \mathbf{N}^{cr^T}\mathbf{T}^T\mathbf{D}^o\mathbf{T}\mathbf{N}^{cr}dV\Delta^i\hat{\mathbf{E}}^{cr} \\ &+\delta\left(\Delta^i\hat{\mathbf{E}}^{cr^T}\right)\int_V \mathbf{N}^{cr^T}\hat{\mathbf{D}}^{cr}\mathbf{N}^{cr}dV\Delta^i\hat{\mathbf{E}}^{cr} - \delta\left(\Delta^i\mathbf{U}^T\right)\int_V \mathbf{N}^T\Delta\mathbf{f}dV \\ &-\delta\left(\Delta^i\mathbf{U}^T\right)\int_S \mathbf{N}^T\Delta\mathbf{t}dS \\ &= 0.\end{aligned} \tag{5.8}$$

Since $\delta\left(\Delta^i\mathbf{U}^T\right)$ and $\delta\left(\Delta^i\hat{\mathbf{E}}^{cr^T}\right)$ are arbitrary, we obtain the element stiffness equation for the i^{th} element is obtained as

$$\begin{bmatrix} \int_V \mathbf{B}^T\mathbf{D}^o\mathbf{B}dV & -\int_V \mathbf{B}^T\mathbf{D}^o\mathbf{T}\mathbf{N}^{cr}dV \\ -\int_V \mathbf{N}^{cr^T}\mathbf{T}^T\mathbf{D}^o\mathbf{B}dV & \int_V \mathbf{N}^{cr^T}\left(\mathbf{D}^{cr}+\mathbf{T}^T\mathbf{D}^o\mathbf{T}\right)\mathbf{N}^{cr}dV \end{bmatrix}\begin{Bmatrix}\Delta^i\mathbf{U} \\ \Delta^i\hat{\mathbf{E}}^{cr}\end{Bmatrix} = \begin{Bmatrix}\int_V \mathbf{N}^T\Delta\mathbf{f}dV + \int_S \mathbf{N}^T\Delta\mathbf{t}dS \\ 0\end{Bmatrix}. \tag{5.9}$$

After assembling all element stiffness equations and applying prescribed displacements and forces, the system stiffness equation can be arranged as

$$\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}\begin{Bmatrix}\Delta\mathbf{U} \\ \Delta\hat{\mathbf{E}}^{cr}\end{Bmatrix} = \begin{Bmatrix}\Delta\mathbf{R}_1 \\ 0\end{Bmatrix} \tag{5.10}$$

where $\Delta\mathbf{U}$ and $\Delta\hat{\mathbf{E}}^{cr}$ are the nodal displacement increment and the nodal local crack strain increment of the system, respectively.

The static condensation is then used to remove the nodal displacement increment from the obtained system stiffness equation. Consequently, the equation can be written in the following form, i.e.,

$$\mathbf{K}^{cr}\Delta\hat{\mathbf{E}}^{cr} = \Delta\mathbf{R}^{cr} \tag{5.11}$$

where $\mathbf{K}^{cr}$ and $\Delta\mathbf{R}^{cr}$ are

$$\mathbf{K}^{cr} = \mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}, \tag{5.12a}$$

$$\Delta\mathbf{R}^{cr} = \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\Delta\mathbf{R}_1. \tag{5.12b}$$

In the consideration of stability of crack patterns, the eigenvalue analysis of $\mathbf{K}^{cr}$ is performed. If all the eigenvalues are positive, it means that the stationary solution in Eq.(5.10) is stable with respect to the current crack pattern. Otherwise, the stationary solution is unstable and bifurcation occurs. Note that this scheme is only used for stability analysis of crack patterns, not for obtaining the displacement solution. The displacement solution will be obtained from the original smeared crack model where the basic unknowns are the nodal displacement increments.

The expression of the total potential energy increment in Eq. (5.1) which is used in the derivation of the mixed finite element formulation is actually the same as the conventional expression used for the conventional smeared crack finite element analysis which is written as

$$\Delta\Pi = \frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}^T\overline{\mathbf{D}}\Delta\boldsymbol{\varepsilon}dV - \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dS - \int_V \Delta\mathbf{u}^T\Delta\mathbf{t}dV. \tag{5.13}$$

To show that Eqs. (5.1) and (5.13) are in fact the same, we substitute Eqs. (3.6) and (3.9) into Eq. (5.1) to obtain

$$\Delta\Pi = \Delta\Pi^M + \Delta\Pi^D$$

$$= \left[\frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}^T \mathbf{D} \Delta\boldsymbol{\varepsilon}\, dV - \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dV - \int_S \Delta\mathbf{u}^T\Delta\mathbf{t}dS\right] + \left[\frac{1}{2}\int_V \Delta\hat{\boldsymbol{\varepsilon}}^{cr^T}\hat{\mathbf{D}}^{cr}\Delta\hat{\boldsymbol{\varepsilon}}^{cr}dV\right] \tag{5.14}$$

By substituting Eq. (3.1) into Eq. (5.14), the equation becomes

$$\left[\int_V {}^T\,{}^o\Delta\, dV - \int_V {}^T\,{}^o\Delta\,{}^{cr}dV - \int_V {}^{cr^T}\,{}^o\Delta\, dV + \int_V {}^{cr^T}\,{}^o\Delta\,{}^{cr}dV + \int_V {}^{cr^T}\,{}^{cr}\,{}^{cr}dV\right] - \int_S {}^T\Delta\, dS - \int_V {}^T\Delta \tag{5.15}$$

Transforming the global crack strain increment $\Delta\boldsymbol{\varepsilon}^{cr}$ to the local crack strain increment $\Delta\hat{\boldsymbol{\varepsilon}}^{cr}$ by applying Eq. (3.3) to Eq. (5.15), we get

$$\Delta\Pi = \frac{1}{2}\begin{bmatrix} \int_V \Delta\boldsymbol{\varepsilon}^T \mathbf{D}^o \Delta\boldsymbol{\varepsilon} dV - \int_V \Delta\boldsymbol{\varepsilon}^T \mathbf{D}^o \mathbf{T} \Delta\hat{\boldsymbol{\varepsilon}}^{cr} dV - \int_V \Delta\hat{\boldsymbol{\varepsilon}}^{cr^T} \mathbf{T}^T \mathbf{D}^o \Delta\boldsymbol{\varepsilon} dV \\ + \Delta\hat{\boldsymbol{\varepsilon}}^{cr^T}\left(\hat{\mathbf{D}}^{cr} + \mathbf{T}^T\mathbf{D}^o\mathbf{T}\right)\Delta\hat{\boldsymbol{\varepsilon}}^{cr} dV \end{bmatrix} - \int_S \Delta\mathbf{u}^T \Delta\mathbf{f} dS - \int_V \Delta\mathbf{u}^T \Delta\mathbf{f} dV. \quad (5.16)$$

Applying Eq. (3.12) to Eq. (5.16), we have

$$\Delta\Pi = \frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}^T\left(\mathbf{D}^o - \mathbf{D}^o\mathbf{T}\left(\hat{\mathbf{D}}^{cr} + \mathbf{T}^T\mathbf{D}^o\mathbf{T}\right)^{-1}\mathbf{T}^T\mathbf{D}^o\right)\Delta\boldsymbol{\varepsilon} dV - \int_S \Delta\mathbf{u}^T \Delta\mathbf{f} dS - \int_V \Delta\mathbf{u}^T \Delta\mathbf{t} dV. \quad (5.17)$$

From Eq.(3.13), the total potential energy increment becomes

$$\begin{aligned} \Delta\Pi &= \frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}^T \overline{\mathbf{D}} \Delta\boldsymbol{\varepsilon} dV - \int_V \Delta\mathbf{u}^T \Delta\mathbf{f} dS - \int_V \Delta\mathbf{u}^T \Delta\mathbf{t} dV \\ &= \frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}^T \Delta\boldsymbol{\sigma} dV - \int_V \Delta\mathbf{u}^T \Delta\mathbf{f} dS - \int_V \Delta\mathbf{u}^T \Delta\mathbf{t} dV \end{aligned} \quad (5.18)$$

which is the same equation as Eq. (5.13).

5.2 Illustrative Model

In order to illustrate the stability analysis method by using the mixed finite element formulation, a simple uniaxial bar shown in Fig. 5.2 is considered. This bar

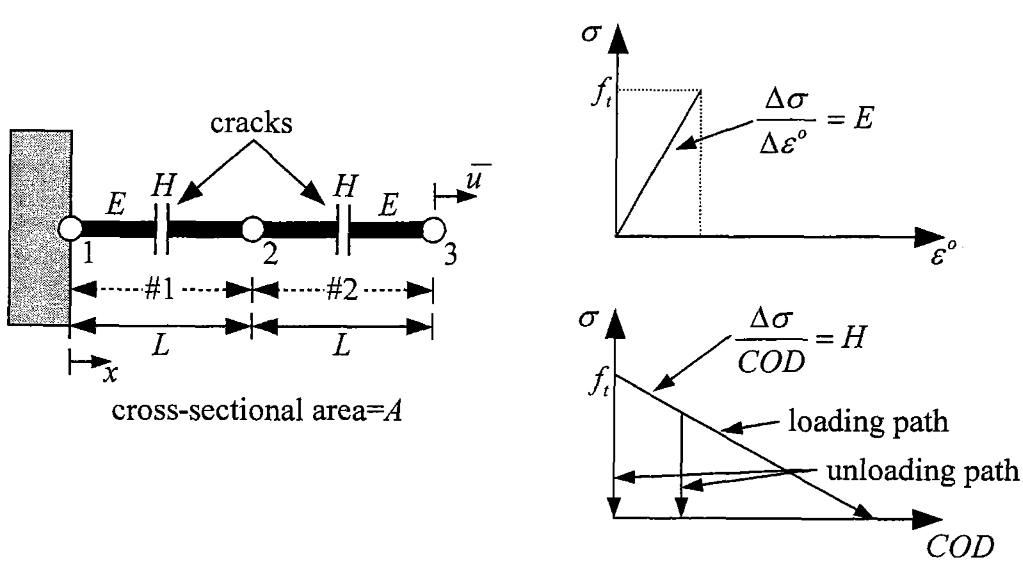
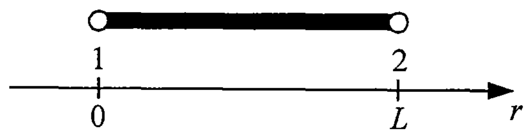


Fig. 5.2 Uniaxial problem using two 1-D elements



$$N_1(r) = \frac{L-r}{L}$$

$$N_2(r) = \frac{r}{L}$$

Fig. 5.3 The shape function for a linear two-noded line element

has one fixed support at one end. At the other end, a controlled displacement $\bar{u}$ is applied. The length of the bar is $2L$ and the cross-sectional area is A. The material is assumed to be elastic with Young's modulus equal to E. The bar is discretized into two elements. Each element has the length equal to L. Each element can accommodate one crack. A crack is assumed to be inactive until the stress in a bar reaches the tensile strength of the material f_t. The characteristic length of each crack, in this case, is equal to the length of the element. The conventional shape function is used for the displacement interpolation (see Fig. 5.3).

Assume that the stress is zero at the beginning. After that, the controlled displacement $\bar{u}$ is increased until the stress in the bar reaches the tensile strength f_t. By the stress criterion for crack initiation, both elements are cracked. Consequently, the cracks follow the constitutive law for cracks (see Fig. 5.2). For opening cracks, a linear relationship between the transmitted tensile stress and the crack opening displacement (COD) with the slope $\frac{\Delta\sigma}{COD}$ equal to H is assumed. This relationship is called the tension-softening relationship. For each unloading crack, a vertical unloading path with constant COD equal to the current COD is applied.

By consider an incremental step after the initiation of cracks, the element stiffness of both elements can be written by using Eq. (5.9) as

$$\mathbf{K} = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & \frac{AE}{2} & \frac{AE}{2} \\ -\frac{AE}{L} & \frac{AE}{L} & -\frac{AE}{2} & -\frac{AE}{2} \\ \frac{AE}{2} & -\frac{AE}{2} & \frac{A(E+\tilde{H})L}{3} & \frac{A(E+\tilde{H})L}{6} \\ \frac{AE}{2} & -\frac{AE}{2} & \frac{A(E+\tilde{H})L}{6} & \frac{A(E+\tilde{H})L}{3} \end{bmatrix} \tag{5.19}$$

where $\tilde{H} = \frac{\Delta\sigma}{\Delta\varepsilon^{cr}} = HL^* = HL$. Here, L^* represents the characteristic length of the crack, which is equal to L for this problem.

After assembling all element stiffness equations, the system stiffness equation is obtained as

$$A\begin{bmatrix} \frac{E}{L} & -\frac{E}{L} & 0 & \frac{E}{2} & \frac{E}{2} & 0 & 0 \\ -\frac{E}{L} & \frac{2E}{L} & -\frac{E}{L} & -\frac{E}{2} & -\frac{E}{2} & \frac{E}{2} & \frac{E}{2} \\ 0 & -\frac{E}{L} & \frac{E}{L} & 0 & 0 & -\frac{E}{2} & -\frac{E}{2} \\ \frac{E}{2} & -\frac{E}{2} & 0 & \frac{(E+\widetilde{H})L}{3} & \frac{(E+\widetilde{H})L}{6} & 0 & 0 \\ \frac{E}{2} & -\frac{E}{2} & 0 & \frac{(E+\widetilde{H})L}{6} & \frac{(E+\widetilde{H})L}{3} & 0 & 0 \\ 0 & \frac{E}{2} & -\frac{E}{2} & 0 & 0 & \frac{(E+\widetilde{H})L}{3} & \frac{(E+\widetilde{H})L}{6} \\ 0 & \frac{E}{2} & -\frac{E}{2} & 0 & 0 & \frac{(E+\widetilde{H})L}{6} & \frac{(E+\widetilde{H})L}{3} \end{bmatrix} \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ \Delta U_3 \\ \Delta^1 \hat{E}_1^{cr} \\ \Delta^1 \hat{E}_2^{cr} \\ \Delta^2 \hat{E}_2^{cr} \\ \Delta^2 \hat{E}_3^{cr} \end{Bmatrix} = \begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \\ \Delta R_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5.20}$$

where ΔU_i and ΔR_i represent the nodal displacement increment and nodal force increment of node i, respectively. Here, $\Delta^i \hat{E}_j^{cr}$ represents the nodal local crack strain increment of node j and, at the same time, of element i. In this problem, ΔU_1, ΔU_3 and ΔR_2 are prescribed. Therefore, the equation can be reduced to

$$A\begin{bmatrix} \frac{2E}{L} & -\frac{E}{L} & -\frac{E}{L} & \frac{E}{L} & \frac{E}{L} \\ -\frac{E}{L} & \frac{(E+\widetilde{H})L}{3} & \frac{(E+\widetilde{H})L}{6} & 0 & 0 \\ -\frac{E}{L} & \frac{(E+\widetilde{H})L}{6} & \frac{(E+\widetilde{H})L}{3} & 0 & 0 \\ \frac{E}{L} & 0 & 0 & \frac{(E+\widetilde{H})L}{3} & \frac{(E+\widetilde{H})L}{6} \\ \frac{E}{L} & 0 & 0 & \frac{(E+\widetilde{H})L}{6} & \frac{(E+\widetilde{H})L}{3} \end{bmatrix} \begin{Bmatrix} \Delta U_2 \\ \Delta^1 \hat{E}_1^{cr} \\ \Delta^1 \hat{E}_2^{cr} \\ \Delta^2 \hat{E}_2^{cr} \\ \Delta^2 \hat{E}_3^{cr} \end{Bmatrix} = AE \begin{Bmatrix} \frac{\Delta \bar{u}}{L} \\ 0 \\ 0 \\ \frac{\Delta \bar{u}}{2} \\ \frac{\Delta \bar{u}}{2} \end{Bmatrix}. \tag{5.21}$$

Performing the static condensation in Eq. (5.12) to remove ΔU_2, we get the stiffness equation in the form of Eq. (5.11) as

$$\frac{AL}{24}\begin{bmatrix} (5E+8\widetilde{H}) & (E+4\widetilde{H}) & 3E & 3E \\ (E+4\widetilde{H}) & (5E+8\widetilde{H}) & 3E & 3E \\ 3E & 3E & (5E+8\widetilde{H}) & (E+4\widetilde{H}) \\ 3E & 3E & (E+4\widetilde{H}) & (5E+8\widetilde{H}) \end{bmatrix} \begin{Bmatrix} \Delta^1 \hat{E}_1^{cr} \\ \Delta^1 \hat{E}_2^{cr} \\ \Delta^2 \hat{E}_2^{cr} \\ \Delta^2 \hat{E}_3^{cr} \end{Bmatrix} = \frac{A}{4} \begin{Bmatrix} E\Delta \bar{u} \\ E\Delta \bar{u} \\ E\Delta \bar{u} \\ E\Delta \bar{u} \end{Bmatrix}. \tag{5.22}$$

The eigenvalues of the obtained stiffness matrix $\mathbf{K}^{cr}$ in the above equation are $\frac{A\widetilde{H}L}{2}$, $\frac{A(E+\widetilde{H})L}{6}$, $\frac{A(E+\widetilde{H})L}{6}$ and $\frac{A(E+\widetilde{H})L}{2}$. It can be seen that all eigenvalues are positive if $\widetilde{H} > 0$. This means that immediately after the two elements are cracked, the equilibrium path is unstable and bifurcation occurs unless both cracks exhibit hardening behavior ($\widetilde{H} > 0$). In reality, cracks will exhibit softening behavior. Therefore, the two cracks cannot continue to open at the same time. One of the cracks must undergo the elastic unloading.

Assume that the crack in element 2 undergoes unloading, the system stiffness equation is changed into

$$A\begin{bmatrix} \frac{E}{L} & -\frac{E}{L} & 0 & \frac{E}{2} & \frac{E}{2} \\ -\frac{E}{L} & \frac{2E}{L} & -\frac{E}{L} & -\frac{E}{2} & -\frac{E}{2} \\ 0 & -\frac{E}{L} & \frac{E}{L} & 0 & 0 \\ \frac{E}{2} & -\frac{E}{2} & 0 & \frac{(E+\widetilde{H})L}{3} & \frac{(E+\widetilde{H})L}{6} \\ \frac{E}{2} & -\frac{E}{2} & 0 & \frac{(E+\widetilde{H})L}{6} & \frac{(E+\widetilde{H})L}{3} \end{bmatrix} \begin{Bmatrix} \Delta U_1 \\ \Delta U_2 \\ \Delta U_3 \\ \Delta^1 \hat{E}_1^{cr} \\ \Delta^1 \hat{E}_2^{cr} \end{Bmatrix} = \begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \\ \Delta R_3 \\ 0 \\ 0 \end{Bmatrix}. \tag{5.23}$$

Following the same process of applying the prescribed boundary conditions and performing the static condensation, we obtain

$$\frac{AL}{24}\begin{bmatrix} (5E+8\widetilde{H}) & (E+4\widetilde{H}) \\ (E+4\widetilde{H}) & (5E+8\widetilde{H}) \end{bmatrix} \begin{Bmatrix} \Delta^1 \hat{E}_1^{cr} \\ \Delta^1 \hat{E}_2^{cr} \end{Bmatrix} = \frac{A}{4}\begin{Bmatrix} E\Delta\overline{u} \\ E\Delta\overline{u} \end{Bmatrix}. \tag{5.24}$$

The eigenvalues of the above stiffness matrix $\mathbf{K}^{cr}$ are $\frac{A(E+\widetilde{H})L}{6}$ and $\frac{A(E+2\widetilde{H})L}{4}$. Both will be positive if $\widetilde{H} > -\frac{E}{2}$. If the crack in element 1 is assumed to undergo elastic unloading instead, the same result will be yielded.

In summary, immediately after the two elements are cracked due to the strength criterion employed, the equilibrium path is unstable and bifurcation occurs unless the two cracks exhibits hardening behavior, i.e., when $H > 0$. In reality, cracks will exhibit softening behavior. Therefore, the two cracks cannot continue to open at the same time. If one of the cracks undergoes the elastic unloading, the stable equilibrium path can be observed as long as $HL > -\frac{E}{2}$. As schematically shown in Fig. 5.4, the cases where there is one opening crack and $HL < -\frac{E}{2}$ present the responses with snapback behavior. Under the displacement control, snapback responses are always unstable.

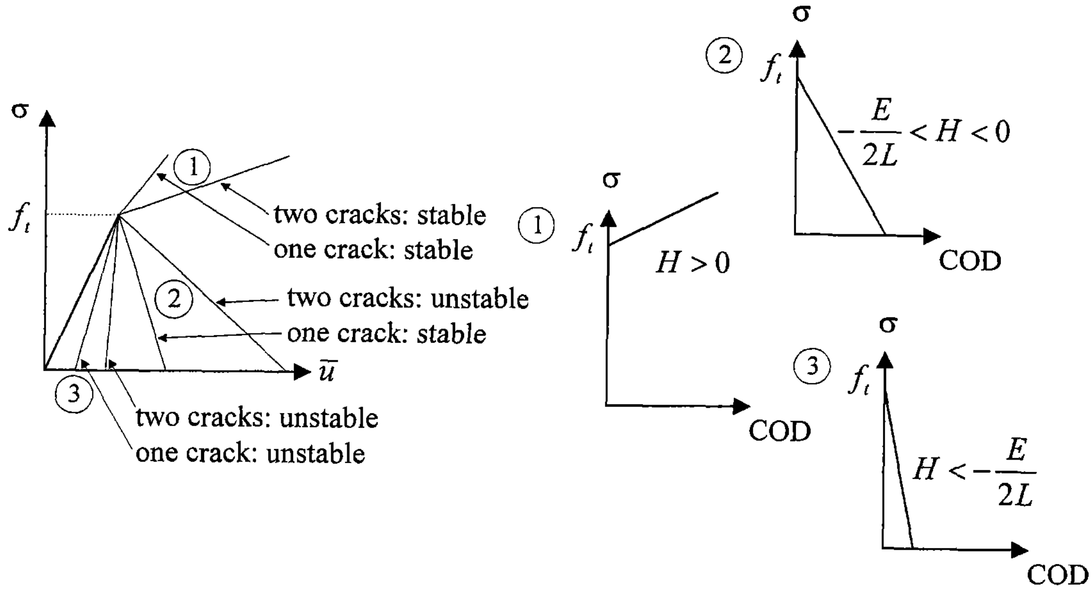


Fig. 5.4 Schematic responses of the uniaxial problem using one-dimensional bar elements