APPENDIX B

Crack Angles

The tension failure of an element occurs when the tensile stress reaches the tensile strength f_t of the material. The orientation of the crack depends on the direction of the principal plane. For 2-D problems, the principal stress σ_p and principal plane θ_p can be calculated by using following equations, i.e.,

$$\sigma_p = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}{}^2}\,, \tag{B.1}$$

$$\theta_p = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}\right) \Big/ 2\,. \tag{B.2}$$

For the current state t, the total stress vector becomes

$$\boldsymbol{\sigma}^t = \left(\sigma_{xx}^t \quad \sigma_{yy}^t \quad \sigma_{xy}^t\right)^T. \tag{B.3}$$

After an incremental step, the total stress vector is defined as

$$\boldsymbol{\sigma}^{t+1} = \left(\sigma_{xx}^{t+1} \quad \sigma_{yy}^{t+1} \quad \sigma_{xy}^{t+1}\right)^T \tag{B.4}$$

where

$$\boldsymbol{\sigma}^{t+1} = \boldsymbol{\sigma}^t + \lambda\Delta\boldsymbol{\sigma}^t \text{ and} \tag{B.5}$$

$$\begin{aligned} \sigma_{xx}^{t+1} &= \sigma_{xx}^t + \lambda\Delta\sigma_{xx}^t \\ \sigma_{yy}^{t+1} &= \sigma_{yy}^t + \lambda\Delta\sigma_{yy}^t \\ \sigma_{xy}^{t+1} &= \sigma_{xy}^t + \lambda\Delta\sigma_{xy}^t. \end{aligned} \tag{B.6}$$

Here, $\Delta\boldsymbol{\sigma}^t$ represents the incremental stress components due to the unit loading applied to this step. In addition, λ represents the load factor that will be applied to this incremental step. For this incremental analysis, each step is stopped when new cracks are initiated or the slope of the tension-softening curve in the existing cracks is changed. For the initiation of new cracks, the new cracks are initiated when the principal stress in solid reaches the tensile strength of the material. Therefore, one of

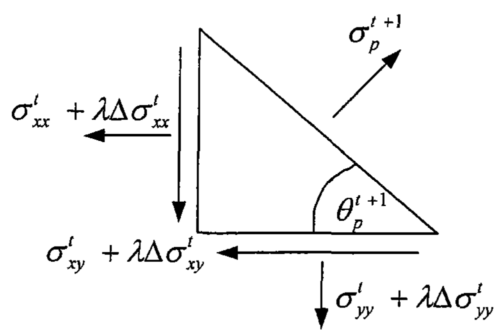

Fig.B.1 Stress components and principal angle the important tasks is to find the multiplication factor λ which makes the principal stress reaches the tensile strength. Note that the value of the multiplication factor λ can be different for different incremental step. As a result, the multiplication factor λ can be written as a function of t, i.e.,

$$\lambda = \lambda(t). \tag{B.7}$$

The calculation begins by setting the principal stress after the incremental step to be equal to the tensile strength, i.e.,

$$f_t = \sigma_p^{t+1}. \tag{B.8}$$

Applying Eq. (B.1) to Eq. (B.8), we have

$$f_t = \frac{\left(\sigma_{xx}^t + \lambda\Delta\sigma_{xx}^t\right) + \left(\sigma_{yy}^t + \lambda\Delta\sigma_{yy}^t\right)}{2} \pm \sqrt{\left[\frac{\left(\sigma_{xx}^t + \lambda\Delta\sigma_{xx}^t\right) - \left(\sigma_{yy}^t + \lambda\Delta\sigma_{yy_y}^t\right)}{2}\right]^2 + \left(\sigma_{xy}^t + \lambda\Delta\sigma_{xy}^t\right)^2}$$

$$= \frac{\sigma_{xx}^t + \sigma_{yy}^t}{2} + \frac{\lambda\left(\Delta\sigma_{xx}^t + \Delta\sigma_{yy}^t\right)}{2} \pm \sqrt{\left[\frac{\sigma_{xx}^t - \sigma_{yy}^t}{2} + \frac{\lambda\left(\Delta\sigma_{xx}^t - \Delta\sigma_{yy}^t\right)}{2}\right]^2 + \left(\sigma_{xy}^t + \lambda\Delta\sigma_{xy}^t\right)^2}. \tag{B.9}$$

Rewrite Eq. (B.9) in the following form, i.e.,

$$f_t = a + \lambda\Delta a \pm \sqrt{(b + \lambda\Delta b)^2 + (c + \lambda\Delta c)^2}, \tag{B.10}$$

where $a = \frac{\sigma_{xx}^t + \sigma_{yy}^t}{2}$, $\Delta a = \frac{\left(\Delta\sigma_{xx}^t + \Delta\sigma_{yy}^t\right)}{2}$, $b = \frac{\sigma_{xx}^t - \sigma_{yy}^t}{2}$, $\Delta b = \frac{\left(\Delta\sigma_{xx}^t - \Delta\sigma_{yy}^t\right)}{2}$, $c = \sigma_{xy}^t$ and $\Delta c = \Delta\sigma_{xy}^t$.

Removing the square root, we get

$$(f_t - a - \lambda\Delta a)^2 = (b + \lambda\Delta b)^2 + (c + \lambda\Delta c)^2, \tag{B.11}$$

which yields

$$\left[\Delta a^2 - \Delta b^2 - \Delta c^2\right]\lambda^2 - 2\left[(f_t - a)(\lambda\Delta a) + b(\lambda\Delta b) + 2c(\lambda\Delta c)\right]\lambda + \left[(f_t - a)^2 - b^2 - c^2\right] = 0 \tag{B.12}$$

Eq. (C.12) can be written as

$$A\lambda^2 + B\lambda + C = 0. \tag{B.13}$$

when

$$\begin{aligned} A &= \Delta a^2 - \Delta b^2 - \Delta c^2 \\ B &= 2[(f_t - a)(\lambda\Delta a) + b(\lambda\Delta b) + 2c(\lambda\Delta c)] \\ C &= (f_t - a)^2 - b^2 - c^2. \end{aligned} \tag{B.14}$$

Consequently, λ can be obtained from

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{B.15}$$

Indeed, there are 2 values of λ from Eq. (B.15). All λ's from all uncracked elements will be investigated. The real λ that will be used in this incremental step will be the one with the smallest absolute value that has the appropriate sign according to the controlled loading parameter.

Finally, the principal angle θ_p is also obtained by using Eq. (B.2) as

$$\theta_p^{t+1} = \tan^{-1}\left(\frac{2\left(\sigma_{xy}^t + \lambda\Delta\sigma_{xy}^t\right)}{\left(\sigma_{xx}^t + \lambda\Delta\sigma_{xx}^t\right) - \left(\sigma_{yy}^t + \lambda\Delta\sigma_{yy}^t\right)} \right) \Big/ 2. \tag{B.16}$$