Chapter 1

Introduction

In [2], T. Leinster defined and studied a class of groups that he called "perfect", in analogy with the notion of "perfect number". In order to do so, he replaced the (multiplicative) arithmetic function $n \mapsto \sum_{d|n} d$ (the sum of the divisors of n) with a suitably defined function D defined, on every finite group G, as the sum of the orders of the normal subgroups of G, namely

$$D : \{\text{groups}\} \longrightarrow \{\text{numbers}\}, \quad G \mapsto \sum_{N \trianglelefteq G} |N|.$$

In the multiplicative theory of numbers a basic role is played by the Dirichlet convolution between arithmetic functions (see, for example, T. Apostol [1]): if f, g are two arithmetic functions, one defines a new arithmetic function $f * g$, called the convolution of f and g, by

$$f * g(n) := \sum_{d|n} f(d)g(n/d),$$

where the sum extends over all positive divisors d of n.

The objective of this work is to make one step further introducing in the new context a suitable notion of convolution. In general, unlike its classical counterpart this operation turns out to be neither commutative nor associative (as shown by explicit examples), however it becomes so when restricted to Abelian groups.

In turn, this is not barely a matter of sharing the same abstract properties, as we show that our convolution can be regarded as a natural extension of the Dirichlet convolution when we consider the naturals embedded in the space of (finite Abelian) groups through the cyclic map (that associates to every natural number n the cyclic group C_n of order n).

On the technical side, as main tools, we use several facts from group theory, as the Jordan-Hölder series, the isomorphism theorems, and duality theory.

In retrospective, following T. Leinster original intuition, the following dictionary could help in figuring out what is going on:

number theory	proposed framework
natural number	finite group
divisor	normal subgroup
quotient	quotient group
prime	simple group
prime decomposition	composition series
product	direct product

The following work is arranged in this way.

Chapter 2 contains all basic definitions, examples and results for multiplication of arithmetic functions, properties of Dirichlet convolution in general group and number theory. Section 2.4 in the same chapter 2, present also extremely detailed review of the definition of perfect groups and the results (Abelian quotient theorem and characterizations of Abelian perfect groups) obtained by T. Leinster [2].

In chapter 3 we deal with the main objective of our work:

- We extend T. Leinster's idea considering, not only complex valued, but "ring-valued functions", defined on the class $\mathcal{G}$ of finite groups and invariant under group isomorphisms. For any such function $\phi : \mathcal{G} \to R$ in the commutative unital associative ring R, we construct another ring-valued function $F_\phi : \mathcal{G} \to R$ (defined as the product of the values of ϕ on all the factors in the composition series of the group) and we show that $DF_\phi : \mathcal{G} \to R$ which is defined as the sum of the values of F_ϕ on all the normal subgroups of the group; is multiplicative i.e. that, if G_1 and G_2 are coprime finite groups, $DF_\phi(G_1 \times G_2) = DF_\phi(G_1) \cdot DF_\phi(G_2)$.

- We generalize the notion of Dirichlet convolution, from arithmetic functions in number theory to our setting, defining Dirichlet convolution for isomorphism-invariant ring-valued functions on finite groups.

- We prove that our generalized Dirichlet convolution preserve the multiplicativity property of functions, i.e., the Dirichlet convolution of multiplicative functions is multiplicative.

- We provide a lot of examples of calculation of Dirichlet convolutions for isomorphism invariant functions on finite groups and we show that, in general, commutativity and associativity are not valid.

- We prove that commutativity and associativity of our Dirichlet convolution are restored if we consider only the case of finite Abelian groups. In more detail, if F_1, F_2, F_3 are ring-valued isomorphism invariant functions defined on finite Abelian groups,

$$F_1 * F_2 = F_2 * F_1,$$
$$(F_1 * F_2) * F_3 = F_1 * (F_2 * F_3).$$

- We finally show that, taking the ring R as the field of complex numbers and restricting our functions to the class of finite cyclic groups, our theory allows to recover the usual number theoretic setting i.e. there is an isomorphism between the family of arithmetic functions with number theoretic Dirichlet convolution and the family of isomorphism invariant complex-valued functions on finite cyclic groups with our Dirichlet convolution.